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(UGC-AUTONOMOUS)

ELECTRICAL TECHNOLOGY

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UNIT-1

TRANSIENT ANALYSIS (FIRST AND SECOND ORDER CIRCUITS)

- **Introduction**
- **Transient Response of RL, RC series and RLC circuits for DC excitations**
- **Initial conditions**
- **Solution using Differential equations approach**
- **Solution using Laplace transformation**
- **Summary of Important formulae and Equations**
- **Illustrative examples**

Introduction:

In this chapter we shall study transient response of the RL, RC series and RLC circuits with external DC excitations. Transients are generated in Electrical circuits due to abrupt changes in the operating conditions when energy storage elements like Inductors or capacitors are present. Transient response is the dynamic response during the initial phase before the steady state response is achieved when such abrupt changes are applied. To obtain the transient response of such circuits we have to solve the differential equations which are the governing equations representing the electrical behavior of the circuit. A circuit having a single energy storage element i.e. either a capacitor or an Inductor is called a Single order circuit and it's governing equation is called a First order Differential Equation. A circuit having both Inductor and a Capacitor is called a Second order Circuit and its governing equation is called a Second order Differential Equation. The variables in these Differential Equations are currents and voltages in the circuit as a function of time.

A solution is said to be obtained to these equations when we have found an expression for the dependent variable that satisfies both the differential equation and the prescribed initial conditions. The solution of the differential equation represents the ***Response*** of the circuit. Now we will find out the response of the basic RL and RC circuits with DC Excitation.

RL **CIRCUIT with external DC excitation:**

Let us take a simple RL networksubjected to external DC excitation as shown in the figure. The circuit consists of a battery whose voltage is ***V*** in serieswith a switch, a resistor ***R,*** and an inductor ***L***. The switch is closed at$t = 0$.

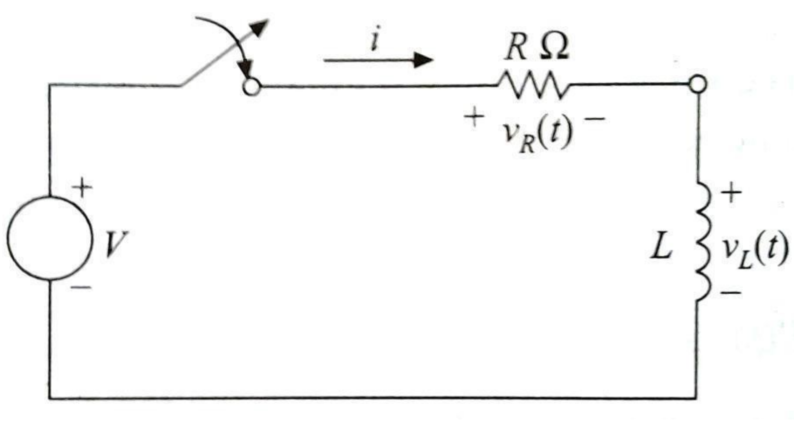


Fig: RL Circuit with external DC excitation

When the switch is closed current tries to change in the inductor and hence a voltage **$V_L(t)$ i**s induced across the terminals of the Inductor in opposition to the applied voltage. The rate of change of current decreases with time which allows current to build up to it's maximum value.

and, hence,

Rearranging we get

It is evident that the current ***i*(*t*)** is zero before ***t* = 0.** and we have to find out current ***i*(*t*)** for time **t >0**. We will find ***i*(*t*)** for time **t >0** by writing the appropriate circuit equationand then solving it by separation of the variables and integration.

Applying Kirchhoff's voltage law to the above circuit we get **:**

$$\mathbf{V = v_R(t) + v_L(t)}$$

$$\text{i } \mathbf{(t) = 0} \text{ for } \mathbf{t < 0} \text{ and}$$

Using the standard relationships of Voltage and Current for the Resistors and Inductors we can rewrite the above equations as

$$\mathbf{V = \quad Ri + Ldi/dt} \text{ for } \mathbf{t > 0}$$

One direct method of solving such a differential equation consists of writing the equation in such a way that the variables are separated, and then integrating each side of the equation. The variables in the above equation are **i** and **t.** Thisequation is multiplied by **dt** andarranged with the variables separated as shown below:

$$Ri.\,dt + Ldi = V.\,dt$$

i.e $Ldi = (V - Ri)dt$

i.e $\dfrac{Ldi}{(V - Ri)} = dt$

Next each side is integrated directly to get :

$$-(L/R)\ln(V - Ri) = t + k$$

Where **k** is the integration constant. In order to evaluate **k**, an initial condition must be invoked. Prior to **t = 0, i (t)** is zero, and thus **i (0−) = 0**. Since the current in an inductor cannot change by a finite amount in zero time without being associated with an infinite voltage, we have **i (0+) = 0**. Setting **i = 0** at **t = 0,** in the above equation we obtain

$$-(L/R)\ln(V) = k$$

$$-L/R[\ln(V - Ri) - \ln V] = t$$

$$\ln[(V - Ri)/V] = -(R/L)t$$

Taking antilogarithm on both sides we get

$$(V - Ri)/V = e^{-Rt/L}$$

From which we can see that

$$i(t) = (V/R) - (V/R)e^{-Rt/L} \text{ for } t > 0$$

Thus, an expression for the response valid for all time **t** would be

$$i(t) = V/R\,[1 - e^{-Rt/L}]$$

This is normally written as:

$$i(t) = V/R\,[1 - e^{-t/\tau}]$$

where **‘τ’** is called the ***time constant*** of the circuit and it’s unit is seconds.

The voltage across the ***resistance*** and the ***Inductor***for **t >0**can be written as :

$$v_R(t) = i(t).R = V\,[1 - e^{-t./\tau}]$$

$$v_L(t) = V - v_R(t) = V - V\,[1 - e^{-t./\tau}] = V\,(e^{-t./\tau})$$

A plot of the current**i(t)** and the voltages **v_R(t) & v_L(t)** is shown in the figure below.

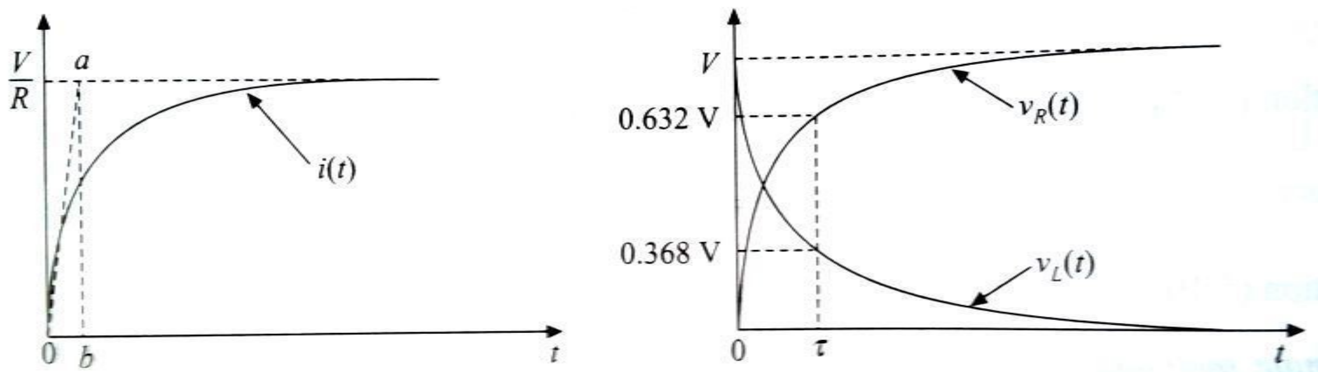


Fig: Transient current and voltages in the Series RL circuit.

At **t** = **'τ'** the voltage across the inductor will be

$$v_L(\tau) = V\,(e^{-\tau/\tau}) = V/e = 0.36788\ V$$

and the voltage across the Resistor will be $v_R(\tau) = V\,[1 - e^{-\tau./\tau}] = 0.63212\ V$

The plots of current**i(t)** and the voltage across the Resistor**v_R(t)** are called ***exponential growth*** curves and the voltage across the inductor**v_L(t)**is called ***exponential decay*** curve.

***RC*CIRCUIT with external DC excitation:**

A series RC circuit with external DC excitation**V** volts connected through a switch is shown in the figure below. If the capacitor is not charged initially i.e. it's voltage is zero ,then after the switch S is closed at time **t=0**, the capacitor voltage builds up gradually and reaches it's steady state value of **V** volts after a finite time. The charging current will be maximum initially (since initially capacitor voltage is zero and voltage acrossa capacitor cannot change instantaneously) and then it will gradually comedown as the capacitor voltagestarts building up. The current and the voltage during such charging periods are called Transient Current and Transient Voltage.

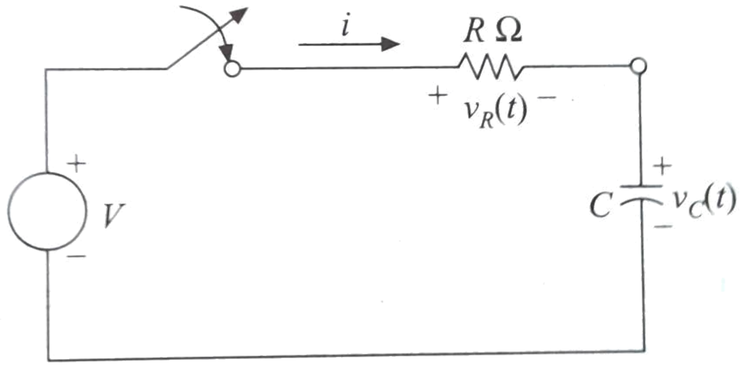


Fig: RC Circuit with external DC excitation

Applying KVL around the loop in the above circuit we can write

$$V = v_R(t) + v_C(t)$$

Using the standard relationships of voltage and current for an Ideal Capacitor we get

$$v_C(t) = (1/C)\int i(t)dt \text{ or } i(t) = C.[dv_C(t)/dt]$$

and using this relation, $v_R(t)$ can be written as $v_R(t) = Ri(t) = R.\ C.[dv_C(t)/dt]$

Using the above two expressions for $v_R(t)$ **and** $v_C(t)$ the above expression for **V** can be rewritten as **:**

$$V = R.\ C.[dv_C(t)/dt] + v_C(t)$$

Or finally $dv_C(t)/dt + (1/RC).\ v_C(t) = V/RC$

The inverse coefficient of $v_C(t)$ *is known as the time constant of the circuit* τ *and is given by* $\tau = RC$ *and it's units are seconds.*

The above equation is a first order differential equation and can be solved by using the same method of ***separation of variables*** as we adopted for the LC circuit.

Multiplying the above equation $dv_C(t)/dt + (1/RC).\ v_C(t) = V/RC$

both sides by '**dt'** and rearranging the terms so as to separate the variables $v_C(t)$ and **t** we get**:**

$$dv_C(t) + (1/RC).\ v_C(t)\ .\ dt = (V/RC).dt$$

dvc(t)

=

[(V/RC)−(1/RC). vc(t)]. dt

dvc(t) / [(V/RC)−(1/RC). vc(t)] = dt

R. C . c(t dt
dvc(t)]
) / =
[(V−v

Now integrating both sides w.r.t their variables i.e. '**vc(t**)' on the LHS and'**t'** on the RHS we get

$$-RC \ln [V - v_C(t)] = t + k$$

where '**k**'is the constant of integration**.**In order to evaluate **k**, an initial condition must be invoked. Prior to **t = 0, vc(t)**is zero, and thus **vc(t)(0−) = 0**. Since the voltage across a capacitor cannot change by a finite amount in zero time, we have **vc(t)(0+) = 0**. Setting **vc(t)= 0** at**t = 0,** in the above equation we obtain**:**

$$-RC \ln [V] = k$$

and substituting this value of **k =** **−RC ln [V]** in the above simplified equation **−RC ln [V − vc(t)] = t+ k** we get **:**

$$-RC \ln [V - v_C(t)] = t - RC \ln [V]$$

i.e. $-RC \ln [V - v_C(t)] + RC \ln [V] = t$

i.e. $-RC [\ln \{V - v_C(t)\} - \ln (V)] = t$

i.e. $[\ln \{V - v_C(t)\}] - \ln [V]\} = -t/RC$

i.e. $\ln [\{V - v_C(t)\}/(V)] = -t/RC$

Taking anti logarithm we get $[\{V - v_C(t)\}/(V)] = e^{-t/RC}$

i.e $v_C(t) = V(1 - e^{-t/RC})$

which is the voltage across the capacitor as a function of time .

The voltage across the Resistor is given by : $v_R(t) = V - v_C(t) = V - V(1 - e^{-t/RC}) = V.e^{-t/RC}$

And the current through the circuit is given by: $i(t) = C.[dv_C(t)/dt] = (CV/CR)e^{-t/RC} = (V/R)e^{-t/RC}$

Or the othe other way: $i(t) = v_R(t)/R = (V.e^{-t/RC})/R = (V/R)e^{-t/RC}$

In terms of the time constant τ the expressions for **vc(t) , vR(t)**and **i(t)** are given by :

$$v_C(t) = V(1 - e^{-t/RC})$$

$$v_R(t) = V.e^{-t/RC}$$

$$i(t) = (V/R)e^{-t/RC}$$

The plots of current **$i(t)$** and the voltages across the resistor **$v_R(t)$** and capacitor **$v_C(t)$** are shown in the figure below.

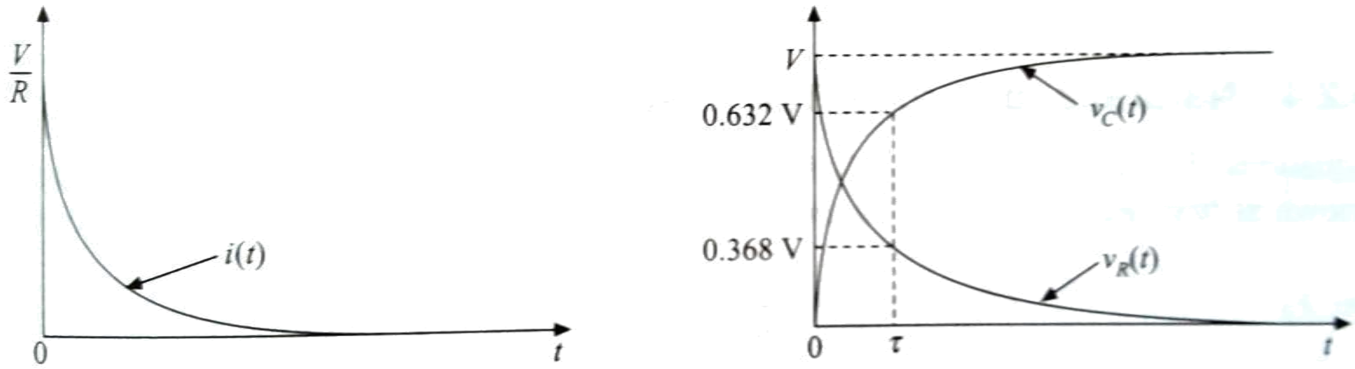


Fig : Transient current and voltages in RC circuit with DC excitation.

At **t** = **'τ'** the voltage across the capacitor will be**:**

$$v_C(\tau) = V\,[1 - e^{-\tau/\tau}] = 0.63212\ V$$

the voltage across the Resistor will be:

$$v_R(\tau) = V\,(e^{-\tau/\tau}) = V/e = 0.36788\ V$$

and the current through the circuit will be:

$$i(\tau) = (V/R)\,(e^{-\tau/\tau}) = V/R.\ e = 0.36788\,(V/R)$$

Thus it can be seen that after one time constant the charging current has decayed to approximately 36.8% of it's value at **t=0** . At **t= 5 τ**charging current will be

$$i(5\tau) = (V/R)\,(e^{-5\tau/\tau}) = V/R.\ e^{5} = 0.0067(V/R)$$

This value is very small compared to the maximum value of **(V/R)** at **t=0** .Thus it can be assumed that the capacitor is fully charged after 5 time constants.

The following similarities may be noted between the equations for the transients in the **LC** and **RC** circuits:

- The transient voltage across the Inductor in a **LC** circuit and the transient current in the **RC** circuit have the same form **k.($e^{-t/\tau}$)**
- The transient current in a **LC** circuit and the transient voltage across the capacitor in the **RC** circuit have the same form **k.($1-e^{-t/\tau}$)**

But the main difference between the **RC and RL** circuits is the effect of resistance on the duration of the transients.

- In a **RL** circuit a large resistance shortens the transient since the time constant τ **=L/R** becomessmall.
- Where as in a **RC** circuit a large resistance prolongs the transient since the time constant τ **= RC** becomes large.

Discharge transients: Consider the circuit shown in the figure below where the switch allows both charging and discharging the capacitor. When the switch is position 1 the capacitor gets charged to the applied voltage V. When the switch is brought to position 2, the current discharges from the positive terminal of the capacitor to the negative terminal through the resistor R as shown in the figure (b). The circuit in position 2 is also called ***source free circuit*** since there is no any applied voltage.

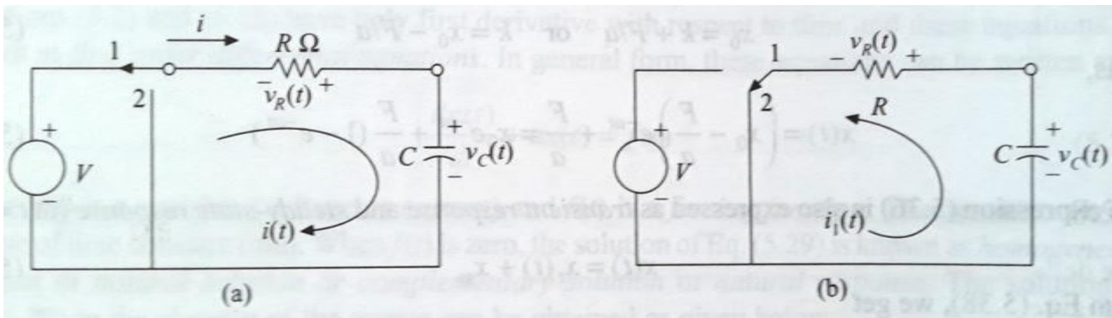


Fig: RC circuit (a) During Charging (b) During Discharging

The current i1 flow is in opposite direction as compared to the flow of the original charging current i. This process is called the *discharging of the capacitor*.The decaying voltage and the current are called the *discharge transients*.The resistor ,during the discharge will oppose the flow of current with the polarity of voltage as shown. Since there is no any external voltage source ,the algebraic sum of the voltages across the Resistance and the capacitor will be zero (applying KVL) .The resulting loop equation during the discharge can be written as

$v_R(t)+v_C(t) = 0$ or $v_R(t) = - v_C(t)$

We know that $v_R(t) =$ R.i(t) = R. C.dv_C(t) /dt. Substituting this in the first loop equation we R.
get C.dv_C(t)/dt + v_C(t) = 0

The solution for this equation is given by vC(t) = $Ke^{-t/\tau}$ where K is a constant decided by the initial conditions and τ =RC is the time constant of the RC circuit

The value of K is found out by invoking the initial condition vC(t) = V @t = 0

Then we get K = V and hence **vC(t) = $Ve^{-t/\tau}$; v_R(t) = $-Ve^{-t/\tau}$ and i(t) = v_R(t)/R = (-V/R)$e^{-t/\tau}$**

The plots of the voltages across the Resistor and the Capacitor are shown in the figure below.

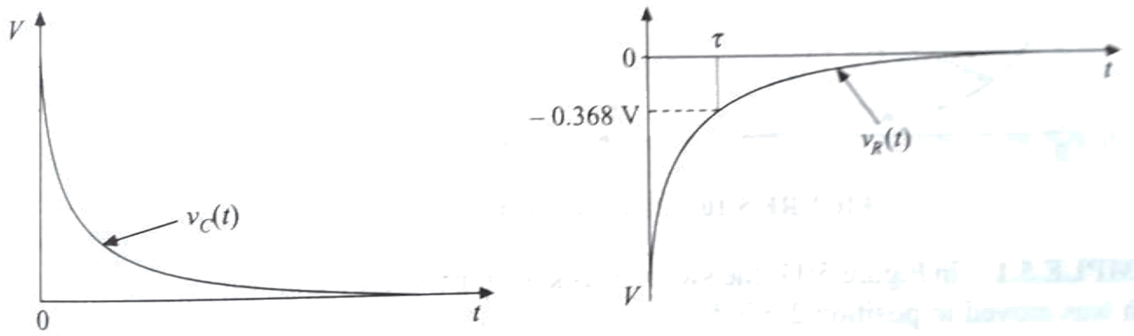


Fig: Plot of Discharge transients in RC circuit

Decay transients: Consider the circuit shown in the figure below where the switch allows both growing and decaying of current through the Inductance . When the switch is position 1 the current through the Inductance builds up to the steady state value of V/R. When the switch is brought to position 2, the current decays gradually from V/R to zero. The circuit in position 2 is also called a ***source free circuit***since there is no any applied voltage.

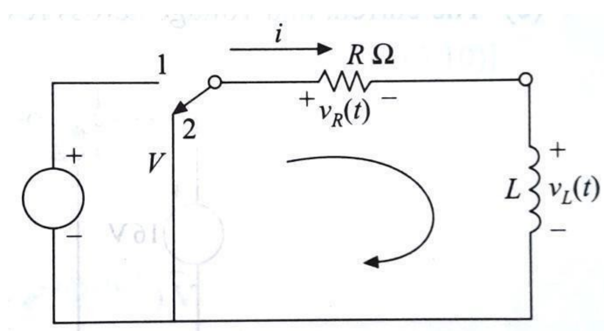


Fig: Decay Transient In RL circuit

The current flow during decay is in the same direction as compared to the flow of the original growing /build up current. The decaying voltage across the Resistor and the current are called the *decay transients*.. Since there is no any external voltage source ,the algebraic sum of the voltages across the Resistance and the Inductor will be zero (applying KVL) .The resulting loop equation during the discharge can be written as

$$v_R(t)+v_L(t) = R.i(t) + L.di(t)/dt = 0 \text{ and } v_R(t) = - v_L(t)$$

The solution for this equation is given by $i(t) = Ke^{-t/\tau}$ where K is a constant decided by the initial conditions and $\tau = L/R$ is the time constant of the RL circuit.

The value of the constant K is found out by invoking the initial condition i(t) = V/R @t = 0

Then we get K = V/R and hence $\mathbf{i(t) = (V/R) \cdot e^{-t/\tau}}$; $\mathbf{v_R(t) = R.i(t) = Ve^{-t/\tau}}$

and $\mathbf{v_L(t) = - Ve^{-t/\tau}}$

The plots of the voltages across the Resistor and the Inductor and the decaying current through the circuit are shown in the figure below.

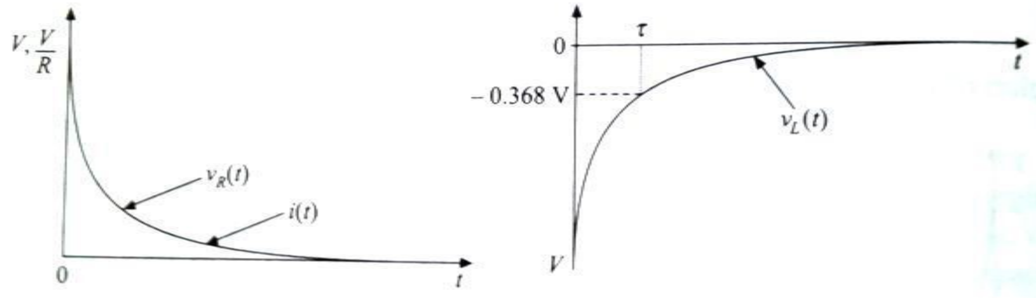


Fig: Plot of Decay transients in RL circuit

The Concept of Natural Response and forced response:

The **RL** and **RC** circuits we have studied are with external DC excitation. These circuits without the external DC excitation are called ***source free circuits*** and their Response obtained by solving the corresponding differential equations is known by many names. Since this response depends on the general ***nature*** of the circuit (type of elements, their size, their interconnection method etc.,) it is often called a ***Natural response***. However any real circuit we construct cannot store energy forever. The resistances intrinsically associated with Inductances and Capacitors will eventually dissipate the stored energy into heat. The response eventually dies down,. Hence it is also called **Transient response**. As per the mathematician's nomenclature the solution of such a homogeneous linear differential equation is called **Complementary function**.

When we consider independent sources acting on a circuit, part of the response will resemble the nature of the particular source. (Or forcing function) This part of the response is called **particular solution. , the steady state response** or **forced response**. This will be complemented by the complementary function produced in the source free circuit. The complete response of the circuit is given by the sum of the **complementary function** and the **particular solution**. In other words:

TheComplete response = Natural response + Forced response

There is also an excellent mathematical reason for considering the complete response to be composed of two parts—the ***forced response*** and the ***natural response***. The reason is based on the fact that the solution of any linear differential equation may be expressed as the sum of two parts: the ***complementarysolution***(natural response) and **the *particular solution***(forced response).

Determination of the Complete Response:

Let us use the same **RL**series circuit with external DC excitation to illustrate how to determine the complete response by the addition of the natural and forced responses. The circuit shown in the figure

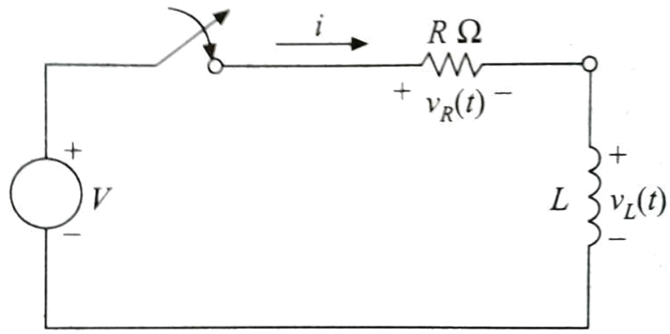


Fig: RL circuit with external DC excitation

was analyzed earlier, but by a different method. The desired response is the current ***i (t),*** and now we first express this current as the sum of the natural and the forced current,

$$\mathbf{i = i_n + i_f}$$

The functional form of the natural response must be the same as that obtainedwithout any sources. We therefore replace the step-voltage source by a short circuit and call it the ***RL source free*** series loop. And $\mathbf{i_n}$ can be shown to be :

$$\boldsymbol{i_n = Ae^{-Rt/L}}$$

where the amplitude ***A***is yet to be determined; since the initial conditionapplies to the***complete*** response, we cannot simply assume ***A = i (0).***We next consider the forced response. In this particular problem theforced response is constant, because the source is a constant ***V***for allpositive values of time. After the natural response has died out, there can beno voltage across the inductor; hence the all ythe applied voltage ***V*** appears across ***R,*** and theforced response is simply

$$\boldsymbol{i_f = V/R}$$

Note that the forced response is determined completely. There is no unknown amplitude. We next combine the two responses to obtain :

$$\mathbf{i = Ae^{-Rt/L} + V/R}$$

And now we have to apply the initial condition to evaluate ***A.*** The current is zero prior to ***t = 0,***and it cannot change value instantaneously since it is the current flowing through an inductor. Thus, the current is zero immediately after ***t = 0***, and

$$\mathbf{A + V/R = 0}$$

So that

$$\mathbf{A = -V/R}$$

And $\mathbf{i = (V/R)(1 - e^{-Rt/L})}$

Note carefully that *A* is not the initial value of **i,** since **A = −V/R,** while **i (0) = 0**.

But In source-free circuits, A would be the initial value of the response given by $\mathbf{i_n = I_0 e^{-Rt/L}}$ (where $\mathbf{I_0}$ =A is the current at time t=0). When forcing functions are present, however,we must first find the initial value of the complete response and then substitute this in the equation for the complete response to find A.Then this value of A is substituted in the expression for the total response i

Amoregeneral solutionapproach:

The method of solving the differential equation by separating the variables or by evaluating the complete response as explained above may not be possible always. In such cases we will rely on a verypowerful method, the success of which will depend upon our intuition or experience. We simply guess or assume a form for the solution and then test our assumptions, first by substitution in the differential equation, and then by applying the given initial conditions. Since we cannot be expected to guess the exact numerical expression for the solution, we will assume a solution containing several unknown constants and select the values for these constants in order to satisfy the differential equation and the initial conditions.

Many of the differential equations encountered in circuit analysis have a solution which may be represented by the exponential function or by the sum of several exponential functions.Hence Let us assume a solution for the following equation corresponding to a source free RL circuit

$$[\,di/dt + (R\,i/L)] = 0$$

in exponential form as

$$i(t) = A.e^{s1t}$$

where **A** and **s1** are constants to be determined. Now substituting this assumed solution in the original governing equation we have:

$$A\,.\,s1\,.\,e^{s1t} + A\,.e^{s1t}\,.\,R/L = 0$$

Or

$$(s1 + R/L).\,A.e^{s1t} = 0$$

In order to satisfy this equation for all values of time, it is necessary that **A = 0, or s1 = −∞, or s1 = −R/L. But if A = 0 or s1 = −∞,** then everyresponse is zero; neither can be a solution to our problem. Therefore, wemust choose

$$s1 = -R/L$$

And our assumed solution takes on the form:

$$i(t) = A.e^{-Rt/L}$$

The remaining constant must be evaluated by applying the initial condition **i (0) = I_0.** Thus, **A = I_0,** and the final form of the assumed solution is(again):

$$i(t) = I_0.e^{-Rt/L}$$

A Direct Route: The Characteristic Equation:

In fact, there is a more direct route that we can take. To obtain the solution for the first order **DE** we solved **s1 + R/L= 0** which is known as the ***characteristic equation*** and then substituting this value of ***s1=-R/L*** in the assumed solution **i (t) = A.e^{s1t}** which is same in this direct method also. We can obtain the characteristic equation directly from the differential equation, without the need for substitution of our trial solution. Consider the general first-order differential equation:

$$a(d\,f/dt) + bf = 0$$

where **a** and **b** are constants. We substitute **s** for the differentiation operator **d/dt** **in the original differential equation** resulting in

$$a(d\, f/dt) + bf = (as + b)\, f = 0$$

From this we may directly obtain the characteristic equation: $as + b = 0$

which has the single root $s = -b/a$. Hence the solution to our differential equationis then given by :

$$f = A.e^{-bt/a}$$

This basic procedure can be easily extended to second-order differential equations which we will encounter for **RLC** circuits and we will find it useful since adopting the variable separation method is quite complex for solving second order differential equations.

RLC CIRCUITS:

Earlier, we studied circuits which contained only **one** energy storage element, combined with a passive network which partly determined how long it took either the capacitor or the inductor to charge/discharge. The differential equations which resulted from analysis were always first-order. In this chapter, we consider more complex circuits which contain **both** an inductor and a**capacitor.** The result is a **second-order** differential equation for any voltage or current of interest. What we learned earlier is easily extended to the study of these so-called ***RLC*** circuits, although now we need **two** initial conditions to solve each differential equation. There are two types of RLC circuits: ***Parallel RLC circuits*** and ***Series circuits .***Such circuits occur routinely in a wide variety of applications and are very important and hence we will study both these circuits.

Parallel RLC circuit:

Let us first consider the simple parallel RLC circuit with DC excitation as shown in the figure below.

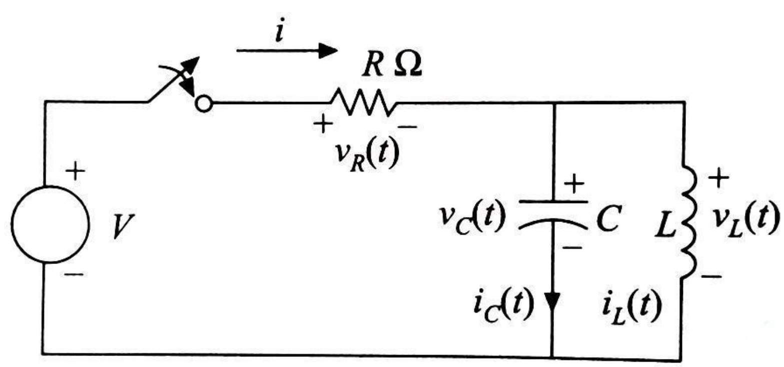


Fig:Parallel *RLC* circuit with DC excitation.

For the sake of simplifying the process of finding the response we shall also assume that the initial current in the inductor and the voltage across the capacitor are zero. Then applying theKirchhoff's current law **(KCL)(** $i = i_C + i_L$ **)**to the common node we get the following integrodifferential equation:

$$(V-v)/R = 1/L\int_{to}^{t} vdt, + C.dv/dt$$

$$V/R = v/R + 1/L\int_{t_0}^{t} v\,dt' + C.dv/dt$$

Where **v = vC(t) = vL(t)** is the variable whose value is to be obtained **.**

When we differentiate both sides of the above equation once with respect to time we get thestandard Linear second-order homogeneous differential equation

$$C.(d^2v/dt^2) + (1/R).(dv/dt) + (1/L).v = 0$$

$$(d^2v/dt^2) + (1/RC).(dv/dt) + (1/LC).v = 0$$

whose solution **v(t)** is the desired response.

This can be written in the form:

$$[s^2 + (1/RC)s + (1/LC)].v(t) = 0$$

where **'s'** is an operator equivalent to **(d/dt)** and the corresponding ***characteristic equation***(as explained earlier as a direct route to obtain the solution) is then given by :

$$[s^2 + (1/RC)s + (1/LC)] = 0$$

This equation is usually called the ***auxiliary equation***or the ***characteristicequation,*** as we discussed earlier .If it can be satisfied, then our assumedsolution is correct. This is a quadratic equation and the roots s_1 and $s_2$are given as :

$$s_1 = -1/2RC + \sqrt{[(1/2RC)^2 - 1/LC]}$$

$$s_2 = -1/2RC - \sqrt{(1/2RC)^2 - 1/LC\,]}$$

And we have the general form of the response as :

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

where s_1 and s_2 are given by the above equations and A_1 and A_2 are two arbitrary constants which are to be selected to satisfy the two specified initial conditions.

Definition of Frequency Terms:

The form of the natural response as given above givesvery little insight intothe nature of the curve we might obtain if ***v(t)***were plotted as a functionof time. The relative amplitudes of $\boldsymbol{A1}$ and $\boldsymbol{A2}$, for example, will certainly beimportant in determining the shape of the response curve. Furthertheconstants $\boldsymbol{s1}$ and $\boldsymbol{s2}$ can be real numbers or conjugate complex numbers,depending upon the values of ***R, L,*** and***C***in the given network. These twocases will produce fundamentally different response forms. Therefore, it willbe helpful to make some simplifying substitutions in the equations for $\boldsymbol{s1}$ and $\boldsymbol{s2}$.Since the exponents $\boldsymbol{s1t}$and $\boldsymbol{s2t}$ must be dimensionless, $\boldsymbol{s1}$ and $\boldsymbol{s2}$ musthave the unit of some dimensionless quantity "per second." Hence in the equations for $\boldsymbol{s1}$ and $\boldsymbol{s2}$ we see that the units of $\boldsymbol{1/2RC}$ and $\boldsymbol{1/\sqrt{LC}}$must also be $\mathbf{s}^{-1}$(i.e., $\mathbf{seconds}^{-1}$). Units of this type are called ***frequencies.***
Now two new terms are defined as below :

$$\omega_0 = 1/\sqrt{LC}$$

which is termed as ***resonant frequency*** and

$$\alpha = 1/2RC$$

which is termed as the ***exponential damping coefficient***

α ***the exponential damping coefficient*** is a measure of how rapidly the natural response decays or damps out to its steady, final value (usually zero). And **s, s_1, and s_2,** are called ***complex frequencies.***

We should note that **s_1, s_2**, **α,** and **ω_0** are merely symbols used to simplify the discussion of ***RLC*** circuits. They are not mysterious new parameters of any kind. It is easier, for example, to say ***"alpha"*** than it is to say ***"the reciprocal of 2RC."***

Now we can summarize these results.

The response of the parallel ***RLC*** circuit is given by :

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad .... \quad [1]$$

where

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad .... \quad [2]$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad .... \quad [3]$$

$$\alpha = 1/2RC \quad .... \quad [4]$$

and

$$\omega_0 = 1/\sqrt{LC} \quad .... \quad [5]$$

A_1 and ***A_2*** must be found by applying the given initial conditions.

We note three basic scenarios possible with the equations for **s_1** and **s_2** depending on the relative values of **α** and **ω_0** (which are in turn dictated by the values of **R, L,** and **C**).

CaseA:

$\alpha > \omega_0$, i.e when $(1/2RC)^2 > 1/LC$ **s_1** and **s_2** will both be negative real numbers, leading to what is referred to as an ***over damped response*** given by :

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

Since **s_1** and **s_2** are both negative real numbers this is the (algebraic) sum of two decreasing exponential terms. Since **s_2** is a larger negative number it decays faster and then the response is dictated by the first term **$A_1e^{s_1t}$.**

CaseB :

$\alpha = \omega_0$, i.e when $(1/2RC)^2 = 1/LC$, **s_1** and **s_2** are equal which leads to what is called a ***critically damped response*** given by **:**

$$v(t) = e^{-\alpha t}(A_1t + A_2)$$

Case C :

$\alpha < \omega_0$, i.e when $(1/2RC)^2 < 1/LC$ both s_1 and s_2 will have nonzero imaginary components, leading to what is known as an ***under damped response*** given by :

$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

where ω_d is called ***natural resonant frequency*** and is given given by**:**

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

We should also note that the general response given bythe above equations [1] through [5] describe not only the voltage but all three branch currents in the parallel *RLC* circuit; the constants A_1 and A_2 will be different for each, of course.

Transient response of a series RLC circuit:

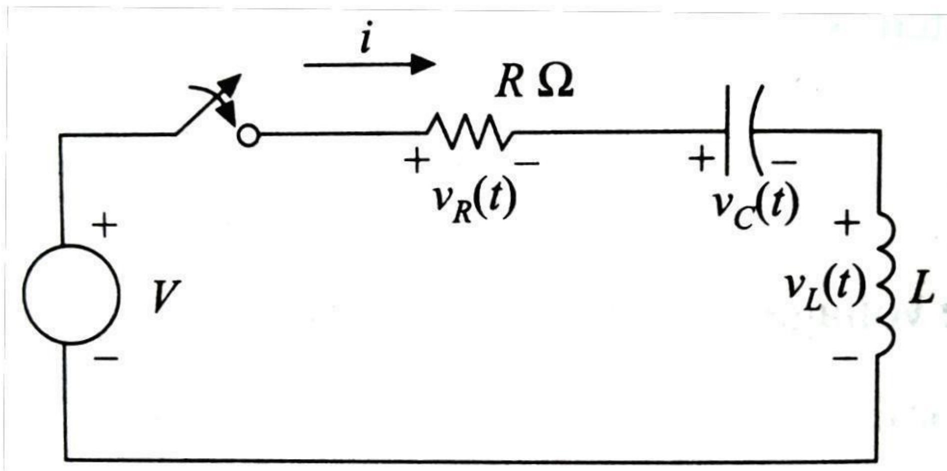


Fig: Series RLC circuit with external DC Excitation

Applying **KVL** to the series **RLC** circuit shown in the figure above at **t= 0** gives the following basic relation :

$$V = v_R(t) + v_C(t) + v_L(t)$$

Representing the above voltages in terms of the current **i** in the circuit we get the following integrodifferentialequation:

$$Ri + 1/C\int i\,dt + L.\,(di/dt) = V$$

To convert it into a differential equation it is differentiated on both sides with respect to time and we get

$$L(d^2i/dt^2) + R(di/dt) + (1/C)i = 0$$

This can be written in the form

$[S^2 + (R/L)s + (1/LC)].i = 0$ where **'s'** is an operator equivalent to **(d/dt)**

And the corresponding characteristic equation is then given by

$$[s^2 + (R/L)s + (1/LC)] = 0$$

This is in the standard quadratic equation form and the roots s_1 and s_2 are given by

$$s_1, s_2 = -R/2L \pm \sqrt{[(R/2L)^2 - (1/LC)]} = -\alpha \pm \sqrt{(\alpha^2 - \omega_0^2)}$$

Where **α** is known as the same ***exponential damping coefficient*** and ω_0 is known as the same ***Resonant frequency*** as explained in the case of Parallel RLC circuit and are given by **:**

$$\alpha = R/2L \quad \text{and} \quad \omega_0 = 1/\sqrt{LC}$$

and A_1 and A_2 must be found by applying the given initial conditions.

Here also we note three basic scenarios with the equations for **s1** and **s2** depending on the relative sizes of **α** and ω_0 (dictated by the values of **R, L,** and **C**).

CaseA:

$\alpha > \omega_0$, i.e when $(R/2L)^2 > 1/LC$, s_1 and s_2 will both be negative real numbers, leading to what is referred to as an ***over damped response*** given by :

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Since s_1 and s_2 are both be negative real numbers this is the (algebraic) sum of two decreasing exponential terms. Sinc s_2 is a larger negative number it decays faster and then the response is dictated by the first term $A_1 e^{s_1 t}$.

Case B :

$\alpha = \omega_0$, i.e when $(R/2L)^2 = 1/LC$ s_1 and s_2 are equal which leads to what is called a ***critically damped response*** given by **:**

$$i(t) = e^{-\alpha t}(A_1 t + A_2)$$

Case C :

$\alpha < \omega_0$, i.e when $(R/2L)^2 < 1/LC$ both s_1 and s_2 will have nonzero imaginary components, leading to what is known as an ***under damped response*** given by :

$$i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

where ω_d is called ***natural resonant frequency*** and is given given by**:**

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Here the constants A_1 and A_2 have to be calculated out based on the initial conditions case by case**.**

Summary of the Solution Process:

In summary, then, whenever we wish to determine the transient behavior of asimple three-element **RLC**circuit, we must first decide whether it is a series or a parallel circuit, so that we may use the correct relationshipfor **α**. The two equations are

$\alpha = 1/2RC$ **(parallel RLC)**
$\alpha = R/2L$ **(series RLC)**

Our second decision is made after comparing **α**with **ω_0**, which is given foreither circuit by **$\omega_0 = 1/\sqrt{LC}$**

- If**$\alpha > \omega_0$**, the circuit is ***over damped***, and the natural response has the form

$$f_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$s_{1,2} = -\alpha \pm \sqrt{(\alpha^2 - \omega_0^2)}$$

- If **$\alpha = \omega_0$**, then the circuit is ***critically damped***and

$$f_n(t) = e^{-\alpha t}(A_1 t + A_2)$$

- And finally**,** if**$\alpha < \omega_0$,** then we are faced with the ***underdamped***response,

$$f_n(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

where

$$\omega_d = \sqrt{(\omega_0^2 - \alpha^2)}$$

Solution using Laplace transformation method:

In this topic we will study Laplace transformation method of finding solution for the differential equations that govern the circuit behavior. This method involves three steps:

- First the given Differential equation is converted into **"s"** domain by taking it's Laplace transform and an algebraic expression is obtained for the desired variable
- The transformed equation is split into separate terms by using the method of Partial fraction expansion
- Inverse Laplace transform is taken for all the individual terms using the standard inverse transforms.

The expression we get for the variable in time domain is the required solution.

For the ease of reference a table of important transform pairs we use frequently is given below.

Table of Important Transform pairs

$f(t)$ (*Function*)	$F(s)$ (*Laplace Transform*)
$u(t)$ (unit step)	$1/s$
$\delta(t)$ (unit impulse)	1
e^{-at}	$\frac{1}{(s+a)}$
$\sin \omega t$	$\frac{w}{(s^2+\omega^2)}$
$\cos \omega t$	$\frac{s}{(s^2+\omega^2)}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
t	$1/s^2$
$\frac{df(t)}{dt}$	$sF(s)$
$\int f(t)dt$	$F(s)/s$

This method is relatively simpler compared to Solving the Differential equations especially for higher order differential equations since we need to handle only algebraic equations in **'s'** domain. This method is illustrated below for the series **RL,RC** and **RLC** circuits.

Series RL circuit with DC excitation:

Let us take the ***series RL*** circuit with external DC excitation shown in the figure below.

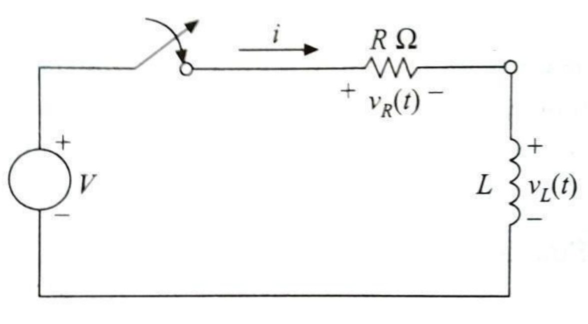


Fig: RL Circuit with external DC excitation

The governing equation is same as what we obtained earlier.

V = Ri + Ldi/dt for **t >0**

Taking Laplace transform of the above equation using the standard transform functions we get

$$V/s = R.I(s) + L[sI(s) - i(0)]$$

It may be noted here that **i(0)** is the initial value of the current at t=0 and since in our case at t=0 just when the switch is closed it is zero , the above equation becomes:

$$V/s = R.I(s) + L[sI(s)] = I(s)[R + L.s]$$

Or

$$I(s) = \left[\frac{(\frac{V}{L})}{s\{s+\frac{R}{L}\}}\right] = \frac{A}{s} + \frac{B}{[s+\frac{R}{L}]}$$ (Expressing in the form of Partial fractions)

Where $A = \left[\frac{(\frac{V}{L})}{\{s+\frac{R}{L}\}}\right]_{s=0} = V/R$ and $B = \left[\frac{(\frac{V}{L})}{s}\right]_{s=-(R/L)} = -V/R$

Now substituting these values of **A** and **B** in the expression for $I(s) = \frac{A}{s} + \frac{B}{[s+\frac{R}{L}]}$ we get

$$I(s) = \frac{V/R}{s} - \frac{V/R}{[s+\frac{R}{L}]}$$

Taking inverse transform of the above expression for **I(s)** using the standard transform pairs we get the solution for **i(t) as**

$$i(t) = (V/R) - (V/R).e^{-(R/L)t} = (V/R)(1 - e^{-(R/L)t})$$

Which is the same as what we got earlier by solving the governing differential equation directly.

RC Circuit with external DC excitation:

Let us now take the ***series RC*** circuit with external DC excitation shown in the figure below.

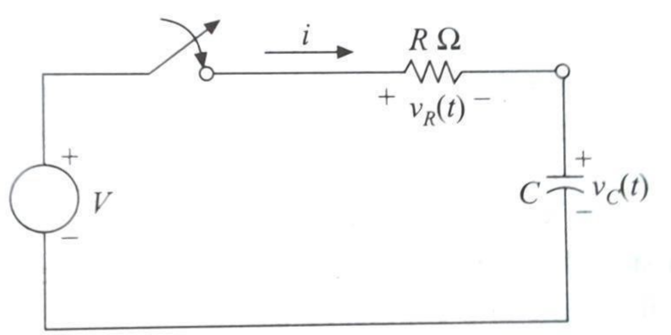


Fig: RC Circuit with external DC excitation

The governing equation is same as what we obtained earlier and is worked out again for easy understanding :

Applying KVL around the loop in the above circuit we can write:

$$V = v_R(t) + v_C(t)$$

Using the standard relationships of voltage and current for an Ideal Capacitor we get

$$v_C(t) = (1/C) \int i(t)dt \text{ or } i(t) = C.[dv_C(t)/dt]$$

(Assuming that the initial voltage across the capacitor $v_c(0) = 0$)

and using this relation, $v_R(t)$ can be written as $v_R(t) = Ri(t) = R.\,C.[d\,v_C(t)/dt]$

Using the above two expressions for $v_R(t)$ **and** $v_C(t)$ the above expression for **V** can be rewritten as **:**

$$V = R.\,C.[d\,v_C(t)/dt] + v_C(t)$$

Now we will take Laplace transform of the above equation using the standard Transform pairs and rules:

$$V/s \quad = R.C.s.v_c(s) + v_c(s)$$

$$V/s \quad = v_c(s)\,(R.C.s.+ 1)$$

$$v_C(s) = (V/s)/(R.C.s + 1)$$

$$v_C(s) = (V/RC)/[s.\,(s + 1/RC)]$$

Now expanding this equation into partial fractions we get

$$v_C(s) = (V/RC)/[s.\,(s + 1/RC)] = A/s + B/(s + 1/RC) \text{ ----(1)}$$

Where $A = (V/RC)/(1/RC)] = V$ and $B = (V/RC)/-(1/RC)] = -V$

Substituting these values of **A and B** into the above equation (1) for $v_c(s)$ we get

$$v_C(s) = (V/s) - [V/(s + 1/RC)] = V\,[(1/s) - \{1/(s + 1/RC)\}]$$

And now taking the inverse Laplace transform of the above equation we get

$$v_C(t) = V(1 - e^{-t/RC})$$

which is the voltage across the capacitor as a function of time and is the same as what we obtained earlier by directly solving the differential equation.

And the voltage across the Resistor is given by $v_R(t) = V - v_C(t) = V - V(1 - e^{-t/RC})$ $= V.e^{-t/RC}$

And the current through the circuit is given by **i(t) = C.[dvc(t)/dt] = (CV/RC)e** $^{-t/RC}$ **=(V/R)e** $^{-t/RC}$

Series RLC circuit with DC excitation:

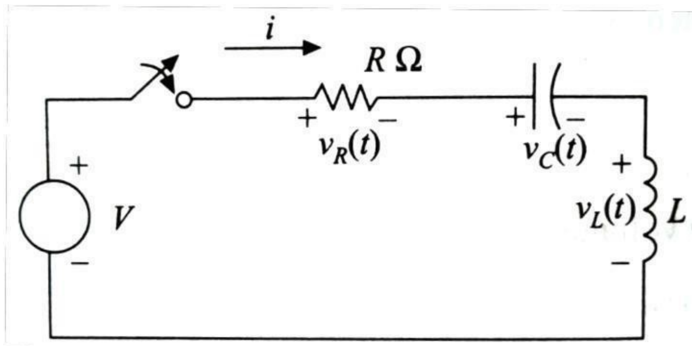


Fig: Series RLC circuit with DC excitation

The current through the circuit in the Laplace domain is given by :

$$\mathbf{I(s)} = \frac{(V/s)}{(R + Ls + 1/Cs)}$$

[since **L [V]** = **V/s**and the Laplace equivalent of the series circuit is given by **Z(s) = (R + Ls + 1/Cs)]**

$$= V/(Rs + Ls^2 + 1/C) = (V/L)/[s^2 + (R/L)s + 1/LC] = \frac{(V/L)}{(s+a)(s+b)}$$

Where the roots **'a'** and **'b'** are given by

$$a = -R/2L + \sqrt{(R/2L)^2 - 1/LC} \quad \text{and} \quad b = -R/2L - \sqrt{(R/2L)^2 - 1/LC}$$

It may be noted that there are three possible solutions for for **I(s)** and we will consider them. **Case A:** Both **a**and **b** are real and not equal i.e.

(R/2L) > 1/√ LC

Then **I(s)** can be expressed as **I(s) =** $\frac{(V/L)}{\quad} = \frac{K1}{\quad} + \frac{K2}{\quad}$

Where**K1 = [** $\frac{\quad}{\quad}$ **] s= – a** = $\frac{\quad}{\quad}$

(s+a)(s+b) (s+a) (s+b)

(V/L) (V/L)
(s+b) (b a)

Where **K2** = $\left[\frac{(V/L)}{(s+a)}\right]_{s=-b} = \frac{(V/L)}{(a-b)}$

Substituting these values of **K1** and **K2** in the expression for **I(s)** we get :

$$I(s) = \frac{(V/L)}{(s+a)(s+b)} = \frac{(V/L)}{(b-a)}\frac{1}{(s+a)} + \frac{(V/L)}{(a-b)}\frac{1}{(s+b)}$$ and

$$i(t) = \frac{(V/L)}{(b-a)} e^{-at} + \frac{(V/L)}{(a-b)} e^{-bt}$$

Case B : Both **a** and **b** are real and equal i.e. **(a=b=c)** **i.e. (R/2L) = 1/$\sqrt{LC}$**

I(s) =(V/L)/ $(s+c)^2$ when **a = b = c** and

i(t) = (V/L). t. e^{-ct}

Case C : Both **a** and **b** are complexconjugates i.e.**a = b*** when **(R/2L) < 1/$\sqrt{LC}$**

Adopting our standard definitions of **α = R/2L ω0 = 1 /√LC**and **ω d=** $\sqrt{(\omega_0^2 - \alpha^2)}$

The roots **a** and **b** are given by **a = α + jωd**and **b = α– jωd**

Then **I(s)** can be expressed as **I(s)** $= \frac{}{(s+\alpha-j\omega d)\ (s+\alpha+j\omega d)} = \frac{K3}{(s+\alpha-j\omega d)} + \frac{K3*}{(s+\alpha+j\omega d)}$

Here**K3** = $(s+\alpha-j\omega d)\,.\,I(s)|_{s=-\alpha+j\omega d} = \frac{(V/L)}{(s+\alpha+j\omega d)}\Big|_{s=-\alpha+j\omega d} = \frac{(V/L)}{2j\omega d}$

Therefore: **K3 =** $\frac{(V/L)}{2j\omega d}$ and **K3* = -** $\frac{(V/L)}{2j\omega d}$

Now substituting these values **K3** and **K3*** in the above expanded equation for **I(s)** we get

$$I(s) = \frac{(V/L)}{2j\omega d}\frac{1}{(s+\alpha-j\omega d)} - \frac{(V/L)}{2j\omega d}\frac{1}{(s+\alpha+j\omega d)}$$

And now taking inverse transform of **I(s)** we get

$$i(t) = \frac{(V/L)}{2j\omega d} e^{-\alpha t}. e^{j\omega d\, t} - \frac{(V/L)}{2j\omega d} e^{-\alpha t}. e^{-j\omega d\, t}$$

$$i(t) = \frac{(V/L)}{\omega d} e^{-\alpha t}\left[(e^{j\omega d\, t} - e^{-j\omega d\, t})/2j\right]$$

$$i(t) = \frac{(V/L)}{\omega_d} e^{-\alpha t} \sin \omega_d t$$

Summary of important formulae and equations:

RL circuit with external DC excitation (Charging Transient) :

- $i(t) = V/R\,[1 - e^{-t/\tau}]$
- $v_L(t) = V\,(e^{-t/\tau})$
- $v_R(t) = i(t).R = V\,[1 - e^{-t/\tau}]$

Source free RL circuit (Decay Transients) :

- $i(t) = (V/R)\,.\,e^{-t/\tau}$; $v_R(t) = R.i(t) = Ve^{-t/\tau}$ and $v_L(t) = -\,Ve^{-t/\tau}$

RC circuit with external DC excitation (Discharge Transients):

- $v_C(t) = V(1 - e^{-t/RC})$
- $v_R(t) = V.\,e^{-t/RC}$
- $i(t) = (V/R)\,e^{-t/RC}$

Source free RC circuit (Discharge transients) :

- $v_C(t) = Ve^{-t/\tau}$; $v_R(t) = -Ve^{-t/\tau}$ and $i(t) = v_R(t)/R = (-V/R)e^{-t/\tau}$

Series RLC circuit: For this circuit three solutions are possible :

1. **$\alpha > \omega_0$,** i.e when **$(R/2L)^2 > 1/LC$** , **s_1** and **s_2** will both be negative real numbers, leading to what is referred to as an ***over damped response*** given by :

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

2. **$\alpha = \omega_0$**, , i.e when **$(R/2L)^2 = 1/LC$** , **s_1** and **s_2** are equal which leads to what is called a ***critically damped response*** given by **:**

$$i(t) = e^{-\alpha t}(A_1t + A_2)$$

3. **$\alpha < \omega_0$,** i.e when **$(R/2L)^2 < 1/LC$** both **s_1** and **s_2** will have nonzero imaginary components, leading to what is known as an ***under damped response*** given by :

$$i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

where **:**

- **α = (R/2L) and is called the exponential damping coefficient $\omega_0 = 1/\sqrt{LC}$ and is called**
- **the resonant frequency**
- **$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ and** is called the ***natural resonant frequency***

Illustrative Examples:

Example 1: Find the current in a series RL circuit having R =2Ω and L = 10H when a DC voltage V of 100V is applied. Find the value of the current 5 secs. after the application of the DC voltage.

Solution: This is a straightforward problem which can be solved by applying the formula.

First let us find out the *Time constant* **τ** of the series LR circuit which is given by **τ = L/R** secs.

$\therefore \quad \tau = 10/2 = 5$ secs

The current in a series LR circuit after the sudden application of a DC voltage is given by :

$\therefore \quad i(t)$ at 5 secs = $100/2\,(1 - e^{-5/5}) = 5\,(1 - e^{-1}) = 50\,(1 - 1/e) = 31.48$

$$\therefore \quad i(t) \text{ at } 5 \text{ secs} = 31.48 \text{ Amps}$$

Example 2: A series RL circuit has R= 25 Ω and L = 5 Henry. A dc voltage V of 100 V is applied to this circuit at t = 0 secs. Find :

(a) The equations for the charging current , and voltage across R & L

(b) The current in the circuit 0.5 secs after the voltage is applied.

(c) The time at which the drops across R and L are equal.

Solution: The solutions for (a) and (b) are straightforward as in the earlier problem. (a)*Time constant* **τ** of the series LR circuit which is given by τ = L/R secs $\therefore \tau = 5/25 = 1/5$ secs

The charging current is given by $i(t) = V/R\,(1 - e^{-t/\tau})$

- ***It is also given by $i(t) = I\,(1 - e^{-t/\tau})$ where I is the final steady state current and is equal to V/R***

$= 100/25\,(1 - e^{-t/(1/5)}) = 4\,(1 - e^{-5t})$ Amps **$i(t) = 4\,(1 - e^{-5t})$ Amps**

- The voltage across R is given by $v_R = i(t).R = V/R\,(1 - e^{-t/\tau}).R = V\,(1 - e^{-t/\tau})$

$$v_R = 100\,(1 - e^{-5t})$$

- The voltage drop across L can be found in two ways.

1. Voltage across Inductor $v_L = L\ di/dt$

2.

But it is easier to find using the second method. ∴ $v_L = 100 - 100\ (1 - e^{-5t})$

$$\mathbf{v_L = 100.\ e^{-5t}}$$

(b) At time t= 0.5 secs $i(t) = 4\ (1 - e^{5t}) = 4\ (1 - e^{-2.5}) = 3.67$ Amps

(c) To find out the time at which the voltages across the Inductor and the Resistor are equal we can equate the expressions for $v_R = 100\ (1 - e^{-5t})$ and $v_L = 100.e^{-5t}$ and solve for t. But the simpler method is, we know that since the applied voltage is 100 V the condition $v_R = v_L$ will also be satisfied when $v_R = v_L = 50$ V. i.e $v_R = 100\ (1 - e^{-5t}) = 50$ volts and $v_L = 100.e^{-5t} = 50$ V. We will solve the second equation **[v_L = 100. e^{-5t} = 50 V]** to get **t** which is easier.

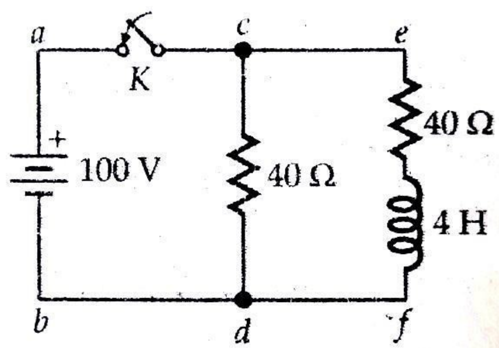
$e^{-5t} = 50/100 = 0.5$.

Taking natural logarithm on both sides we get:

$-5t\ .\ln(e) = \ln 0.5$ i.e $-5t\ .1 = -0.693$ i.e $t = 0.693/5 =$ **0.139 secs**

∴ ***The voltages across the resistance and the Inductance are equal at time t = 0.139 secs***

Example 3: In the figure shown below after the steady state condition is reached , at time t=0 the switch K is suddenly opened. Find the value of the current through the inductor at time t = 0.5 seconds.



Solution: The current in the path ***acdb*** (through the resistance of 40 Ω alone) is 100/40 = 2.5Amps.(Both steady state and transient are same)

The steady state current through the path ***aefb*** (through the resistance of 40 Ω and inductance of 4H) is also = 100/40 = 2.5 Amps.

Now when the switch K is suddenly opened, the current through the path ***acdb***(through the resistance of 40 Ω alone) immediately becomes zero because this path contains only resistance. But the current through the inductor decays gradually but now through the different path ***efdce***

The decay current through a closed RL circuit is given by $I.e^{-t/\tau}$ where I is the earlier steady state current of 2.5 amps through L and $\tau = L/R$ of the decay circuit. ***It is to be noted carefully here that in the decay path both resistors are there and hence R =40+40 = 80Ω***

Hence τ= L/R = 4/80 = 0.05 secs

$-t / 0.01$

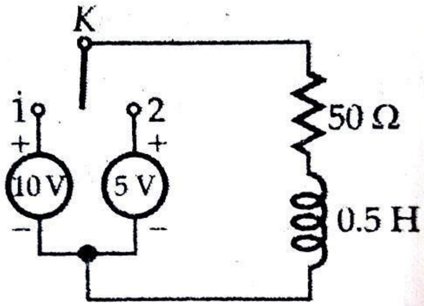
Hence the current through the inductor at time 0.5 secs is given by i(t) @0.5secs $=2.5.e^{-0.5/0.05}$

i.e i(t) @0.5secs = $2.5.e^{-10}$

i.e i(t) @0.5secs= 1.14×10^{-4} Amps

Example 4: In the circuit shown below the switch is closed to position 1 at time t = 0 secs. Then at time t = 0.5 secs the switch is moved to position 2. Find the expressions for the current through the circuit from 0 to 0. 5 msecs and beyond 0. 5 msecs.

Solution: The time constant **τ**of the circuit in both the conditions is same and is given by τ = L/R = 0.5/50 = 0.01 secs



1. During the time t=0 to 0.5 msecs. i(t) is given by the standard expression for growing current through a L R circuit: i(t)during 0 to 0.5 msecs = V/R (1—e $^{-t / τ}$)

And the current i(t) @ t= 0.5 msecs = 10/50 (1-- e $^{– 0.5x10-3 / 0.01}$) = 0.2 (1 – e$^{—0.05}$) = 9.75 mA ***i(t) @ t = 0.5 msecs = 9.75 mA and this would be the initial current when the switch is moved to position 2***

2. During the time beyond 0.5 msecs (switch is in position 2): The initial current is 9.75 mA .

The standard expression for the growing current ***i(t) = V/R (1—e*** $^{-t / τ}$ ***)*** is not applicable now since it has been derived with initial condition of i(t) =0 at t=0 where as the initial condition for the current i(t) now in position 2 is **9.75 mA** . Now an expression for i(t) in position 2 is to be derived from first principles taking fresh t=0 and initial current i(0) as 9.75mA. The governing equation in position 2 is given by :

50i+0.5di/dt = 5

We will use the same *separation of variables method* to solve this differential equation. Dividing the above equation by **0.5**, then multiplying by **dt** and separating the terms containing the two variables **i** and **t** we get:

100i + di/dt = 10 i.e 100i.dt +di = 10.dt i.e di = dt (10 – 100i) i.e di/ (10 – 100i) = dt

Now integrating on both sides we get

--1/100 ln (10 -- 100i) = t + K ----------- (1)

The constant K is now to be evaluated by invoking the new initial condition i(t) = 9.75 mAat t =0

--1/100 ln (10 – 100x9.75X10^{--3}) = K = --1/100 ln (10 – 0.975) = --1/100 ln (9.025)

Substituting this value of K in the above equation (1) we get

--1/100 ln (10 -- 100i) = t --1/100 ln (9.025)

--1/100 ln (10 -- 100i) + 1/100 ln (9.025) = t

--1/100 [ln (10 -- 100i) -- ln (9.025)] = t

--1/100 . ln [(10 -- 100i) / (9.025)] = t

ln [(10 -- 100i) / (9.025)] = --100t

Taking antilogarithm to base **e** on both sides we get:

(10 -- 100i) / (9.025)] = e^{--100t}

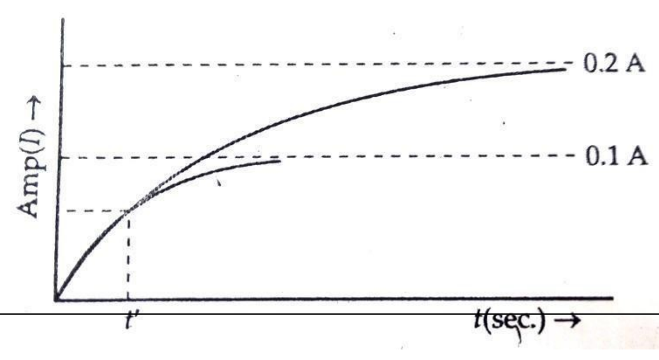
(10 -- 100i) = 9.025 x e^{--100t}

(10 --9.025 x e^{--100t}) = 100i

i = (10 --9.025 x e^{--100t})/ 100 = 10/100 -- 9.025 x e^{--100t} / 100

And finally i = 0.1 –0.09. e^{--100t}

The currents during the periods t = o to 0.5 mses and beyond t = 0.5msec are shown in the figure below. Had the switch been in position 1 all through, the current would have reached the steady state value of 0.2 amps corresponding to source voltage of 10 volts as shown in the top curve. But since the switchis changed to position 2 the current changed it’s path towards the new steady state current of 0.1 Amps corresponding the new source voltage of 5 Volts from 0.5 msecs onwards.



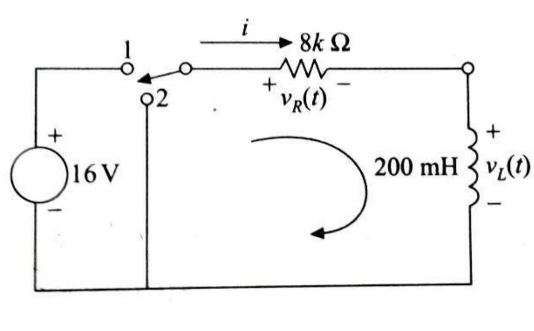
Example 5: In the circuit shown below the switch is kept in position 1 upto 250 μsecs and then moved to position 2. Find

(a) The current and voltage across the resistor at t = 100 μsecs

(b) The current and voltage across the resistor at t = 350 μsecs

(a)
(b.) -

Solution : The time constant **τ** of the circuit is given by τ = L/R = 200mH/8KΩ = 25 μsec and is same in both the switch positions.



(a)The current in the circuit upto 250 μsec (till switch is in position 1) is given by :

$i(t)_{\text{growing}} = V/R\,(1 - e^{-t/\tau}) = (16/8)\text{X}10^{-3}\,(1 - e^{-t/25 \times 10^{-6}}) = 2\text{x}(1 - e^{-t/25 \times 10^{-6}})$ mA

- The current in the circuit @100µsec is given by

  $i(t)_{@100\ \mu sec} = 2\text{x}\,(1 - e^{-100\ \mu sec / 25\ \mu sec})$ mA = $2\text{x}(1 - e^{-4})$ mA = 1.9633 mA

$i(t)_{@100\ \mu sec}$ = 1.9633 m

- The Voltage across the resistoris given by $v_{R@100\ \mu sec} = R \times i(t)_{@100\ \mu sec}$ $v_{R@100\ \mu sec} = 8\ K\Omega \times 1.9633$ mA = 15.707 V

$v_{R@100\ \mu sec}$ = 15.707 V

(b)

- The current in the circuit @350 µsec is the decaying current and is given by:

$i(t)_{\text{Decaying}} = I(0).e$ where I(0) is the initial current and in this case it is the growing current @250µsec. (Since the switch is changed @250µsec) The time t is to be reckoned from this time of 250 µsec. Hence t = (350—250) = 100µsec. So we have to calculate first $i(t)_{\text{growing}(@250\ \mu sec)}$ which is given by:

$i(t)_{\text{growing}(@250\ \mu sec)} = V/R\,(1 - e^{-t/\tau}) = (16/8)\text{X}10^{-3}\,(1 - e^{-t/25\ \mu sec}) = 2\text{x}(1 - e^{-250/25\ \mu sec})$ mA $= 2\text{x}(1 - e^{-10})$ mA = 1.999 mA

$i(t)_{\text{growing}(@250\ \mu sec)}$ = 1.999 mA = I(0)

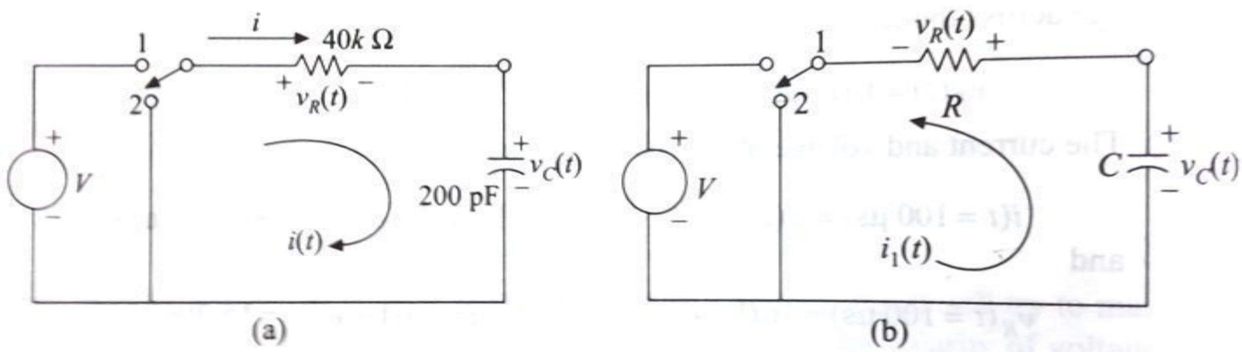
Hence $i(t)_{@350\ \mu sec} = I(0).e^{-t/\tau} = 1.99\text{x}\ e^{-100\ \mu sec/25\ \mu sec}$ mA = $1.99\text{x}\ e^{-4}$ mA = 0.03663 mA **$i(t)_{@350\ \mu sec}$ = 0.03663 mA**

- The voltage across the resistor v_{R} @350 µsec = Rxi(t@350 µsec) = 8KΩx0.03663 mA

$v_{R\ @350\ \mu sec}$= 0.293V

Example 6: In the circuit shown below the switch is kept in position 1 up to 100 µ secs and then it is moved to position 2 . Supply voltage is 5V DC . Find

a) The current and voltage across the capacitor at t = 40 µ secs
b) The current and voltage across the resistor at t = 150 µ secs



Solution: The time constant **τ** of the circuit is same in both conditions and is given by $\tau = RC = 40\times10^{3} \times 200\times10^{-12} = 8$ µsec

a) The time t = 40 µsec corresponds to the switch in position 1 and in that condition the current i(t) is given by the standard expression for charging current

$$\mathbf{i(t) = (V/R)\,[e^{-t/\tau}]}$$

i(t) @40 µsec = $5v/40K\Omega * e^{-40/8}$] Amps = 0.125x[e^{-5}] mA = 0.84224 µA **i(t) @40 µsec = 0.84224 µA**

The voltage across the capacitor during the charging period is given by V [1- $e^{-t/\tau}$].

$v_C(t)$ @40 µsec = 5[1 -- $e^{-40/8}$] = 5[1 -- e^{-5}] = 4.9663 Volts

vC(t) @40 µsec = 4.9663 Volts

b) The time t = 150 µsec corresponds to the switch in position 2 and the current i(t) is given by the discharge voltage expression **i(t) = [$v_C(t)_0$/R]. $e^{-t/\tau}$**

Where **$v_C(t)_0$** is the initial capacitor voltage when the switch was changed to position 2 and it is the voltage that has built up by 100 µsec during the charging time (switch in position 1) and hence is given by

$v_C(t)@100\mu sec = 5[1- e^{-100/8}]$ volts = $5x[1- e^{-12.5}]$ Volts = 4.999 Volts

And now t=150 µsec from beginning is equal to t = (150-100) = 50 µsec from the time switch is changed to position 2.

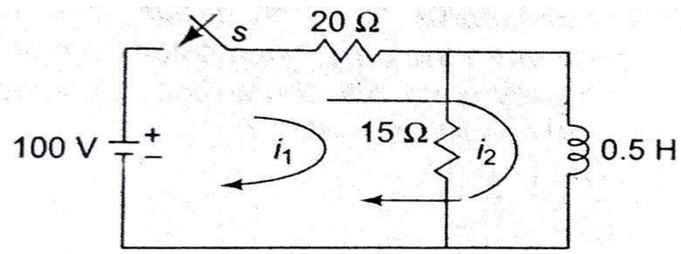
Therefore the current through the resistor at 150 µsec from the beginning = $i(t)_{150\mu sec} = (4.999/40K\Omega).\ e^{-t/\tau}$

$i(t)_{150\mu sec} = 0.1249 \times e^{-50/8} = 0.241\ \mu A$

$i(t)_{150\mu sec}$ = 0.241 µA

And the voltage across the resistor = R x i(t) = 40KΩ x **0.241 µA = 0.00964v**

Example 7: In the circuit shown below find out the expressions for the current i_1 and i_2 when the switch is closed at time t= 0



Solution: It is to be noted that in this circuit there are two current loops 1 and 2 . Current i_1 alone flows through the resistor 15 Ω and the current i_2 alone flows through the inductance0.5

H where as both currents i_1 and i_2 flow through the resistor 20 Ω. Applying KVL to the two loops taking care of this point we get

$20(i_1 + i_2) + 15 i_1 = 100$ i.e $35 i_1 + 20 i_2 = 100$ ------ (1)

and $20(i_1 + i_2) + 0.5\ di_2/dt = 100$; $20 i_1 + 20 i_2 + 0.5\ di_2/dt = 100$ ------- (2)

Substituting the value of $i_1 = [100/35 – (20/35) i_2] = 2.86 – 0.57 i_2$ obtained from the above equation (1) into equation (2) we get :

$20 [2.86 – 0.57 i_2] +20i_2 + 0.5 (di_2/dt) = 100$

$57.14 – 11.4 i_2 +20i_2 + 0.5 (di_2/dt) = 100$

$(di_2/dt)\ i_2 + 17.14\ i_2 = 85.72$

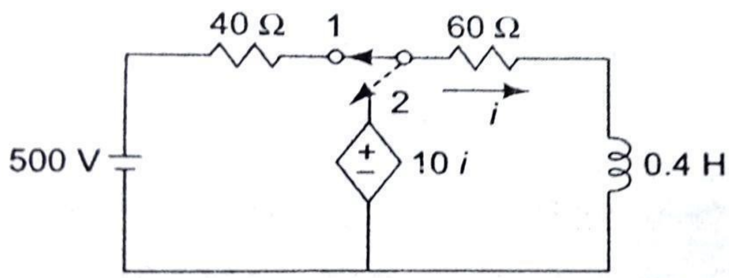
The solution for this equation is given by $i_2(t) = K.\ e^{-17.14t} + 85.72/17.14$ and the constant K can be evaluated by invoking the initial condition. The initial current through the inductor = 0 at time t = 0 .

Hence K = -- 85.72/17.14 = -- 5

Therefore **$i_2(t) = 5\ (1 - e^{-17.14t})$ Amps**

And current $i_1(t) = 2.86 - 0.57\ i_2 = 2.86 - 0.57\ [5\ (1 - e^{-17.14t})] =$ $0.01 + 2.85\ e^{-17.14t}$ Amps And current **$i_1(t) = 0.01 + 2.85\ e^{-17.14t}$ Amps**

Example 8 : In the circuit shown below find an expression for the current i(t) when the switch is changed from position 1 to 2 at time t= 0 .



Solution: The following points are to be noted with reference to this circuit:

- When the switch is changed to position 2 the circuit is equivalent to a normal source free circuit but with a current dependent voltage source given as 10i.
- The initial current in position 2 is same as the current when the switch was in position 1 (for a long time) and is given by $I_0 = 500/(40+60) = 5$ Amps

The loop equation in position 2 is given by : 60i + 0.4 di/dt = 10i i.e (50/0.4) i + di/dt = 0

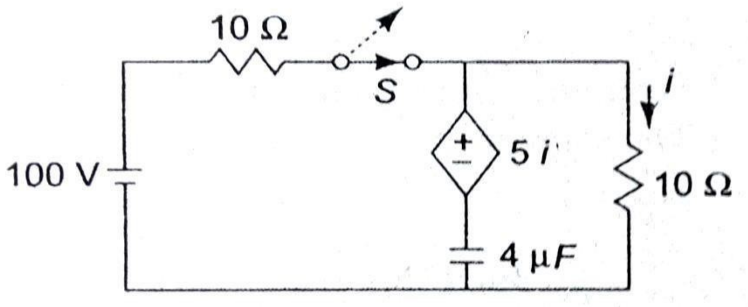
Writing the equation in the **'s'**notation where **'s'** is the operator equivalent to (d/dt) we get

(s+ 125) i = 0 and the characteristic equation will be (s+ 125) = 0

Hence the solution i(t) is given by $i(t) = K\ .\ e^{-125t}$. The constant K can be evaluated by invoking the initial condition that i(t) @ t=0 is equal to I_0 = 5 amps .Then the above equation becomes:

$5 = K\ .\ e^{-125X0}$ i.e K = 5 and hence the current in the circuit when the switch is changed to position 2 becomes: **$i(t) = 5 . e^{--125t}$ Amps**

Example 9 : In the circuit shown below find an expression for the current i(t) when the switch is opened at time t= 0



Solution: The following points may be noted with reference to this circuit:

- When the switch is opened the circuit is equivalent to a normal source free circuit but with a current dependent voltage source given as 5i.
- The initial current I_0 when the switch is opened is same as the current when the switch was closed for a long time and is given by I_0 = 100/(10+10) = 5 Amps

The loop equation when the switch is opened is given by :

$$(1/4\text{x}10^{-6})\int i dt + 10i = 5i$$

$$(1/4\text{x}10^{-6})\int i dt + 5i = 0$$

Differentiating the above equation we get :

$$5.(di/dt) + (1/4\text{x}10^{-6})i = 0 \quad \text{i.e.} = (di/dt) + (1/20 \text{ x } 10^{-6})i = 0$$

Writing the above equation in the **'s'**notation where **'s'** is the operator equivalent to (d/dt) we get

$(s+ 1/20 \text{ x } 10^{-6})$ i = 0 and the characteristic equation will be $(s+ 1/20 \text{ x } 10^{-6}) = 0$

The solution i(t) is given by $i(t) = K . e^{-t/20 \text{ x } 10^{-6}}$. The constant K can be evaluated by invoking the initial condition that i(t) $_{@\, t=0}$ is equal to I_0 = 5 amps .Then the above equation becomes:

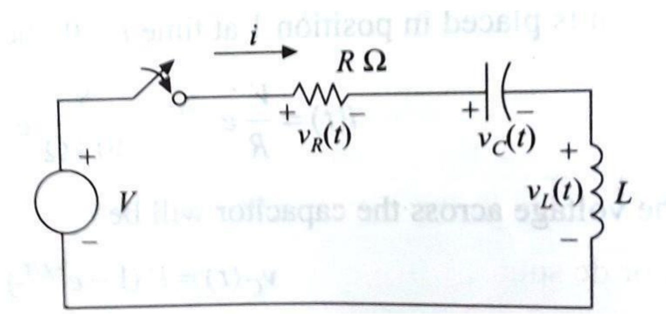
$5 = K . e^{-t/20 \text{ x } 10^{-6}}$ i.e K = 5 and hence the current in the circuit when the switch is opened

becomes: $\quad$ **$i(t) = 5. e^{-t/20 \text{ x } 10^{-6}}$ Amps**

Example 10: A series RLC circuit as shown in the figure below has R = 5Ω,L= 2H and C = 0.5F.The supply voltage is 10 V DC . Find

a) The current in the circuit when there is no initial charge on the capacitor.
b) The current in the circuit when the capacitor has initial voltage of 5V
c) Repeat question (a) when the resistance is changed to 4 Ω

d) Repeat question (a) when the resistance is changed to 1 Ω



Solution: The basic governing equation of this series circuit is given by :

$$Ri + 1/C\int idt + L.(di/dt) = V$$

On differentiation we get the same equation in the standard differential equation form

$$L(d^2i/dt^2) + R(di/dt) + (1/C)i = 0$$

By dividing the equation by L and using the operator '**s' for d/dt** we get theequation in the form of characteristic equation as :

$$[s^2 + (R/L)s + (1/LC)] = 0$$

Whose roots are given by:

$$s_1, s_2 = -R/2L \pm \sqrt{[(R/2L)^2 - (1/LC)]} = -\alpha \pm \sqrt{(\alpha^2 - \omega_0^2)}$$

and three types of solutions are possible.

1. **$\alpha > \omega_0$,** i.e when **$LC > (2L/R)^2$** **s_1** and **s_2** will both be negative real numbers, leading to what is referred to as an ***over damped response*** given by :

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

2. **$\alpha = \omega_0$**, , i.e when **$LC = (2L/R)^2$** **s_1** and **s_2** are equal which leads to what is called a ***critically damped response*** given by **:**

$$i(t) = e^{-\alpha t}(A_1t + A_2)$$

3. **$\alpha < \omega_0$,** i.e when **$LC < (2L/R)^2$** both **s_1** and **s_2** will have nonzero imaginary components, leading to what is known as an ***under damped response*** given by :

$$i(t) = e^{-\alpha t}(A_1\cos\omega_d t + A_2\sin\omega_d t)$$

where ω_d is called ***natural resonant frequency*** and is given given by**:**

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

The procedure to evaluate the complete solution consists of the following steps for each part of the question:

1. We have to first calculate the roots for each part of the question and depending on to which case the roots belong we have to take the appropriate solution .
2. Then by invoking the first initial condition i.e i = 0 at t=0 obtain the first relation between A_1 and A_2 or one of its values.
3. If one constant value is obtained directly substitute it into the above solution and take its first derivative. Or else directly take the first derivative of the above solution

4. Now obtain the value di/dt @ t= 0from the basic RLC circuit equation by invoking the initial conditions of v_C@ t=0 and i(t) @ t=0 . Now equate this to the differential of the solution for i(t) to get the second relation between A_1 and A_2(or the second constant . Now using these two equations we can solve for A_1 and A_2 and subsititute in the solution for i(t) to get the final solution.

(a) $s_1,s_2 = -R/2L \pm \sqrt{[(R/2L)^2 - (1/LC)]} = (-5/2x2) \pm \sqrt{[(5/2x2)^2 - (1/2x0.5)]} = -1.25 \pm 0.75$.

i.e. s_1 = --0.5 and s_2 = --2

In this case the roots are negative real numbers and the solution is given by :

$i(t) = A_1e^{s1t} + A_2e^{s2t} = A_1e^{-0.5t} + A_2e^{-2t}$ ------- (1)

Now we will apply the first initial condition i.e i(t) = 0 at t=0 .Then we get

$0 = A_1e^{-0.5x0} + A_2e^{-2x0}$ i.e. $A_1 + A_2 = 0$

The basic equation for voltage in the series RLC circuit is given as :

$V = R.i(t) + v_C(t) + L.(di/dt)$ i.e $di/dt = 1/L [V - R.i(t) - v_C(t)$

At time t=0 we get

$(di/dt)_{@ t=0} = 1/L [V - R.i(t=0) - v_C(t=0)]$ ----------- (2)

But we know that the voltage across the capacitor and current are zero at time t=0 .

Therefore $(di/dt)_{@ t=0} = V/L = 10/2 = 5$ --------- (3)

Now the equation for i(t) at equation (1) is differentiated to get

$(di/dt) = -0.5A_1e^{-0.5t} - 2A_2e^{-2t}$

and the above value of $(di/dt)_{@ t=0}$ = 5 is substituted in that to get the second equation with A_1 and A_2

$(di/dt)_{@ t=0} = 5 = -0.5A_1e^{-0.5x0} - 2A_2e^{-2x0} = -0.5A_1 - -2A_2$

Now we can solve the two equations for A_1 and A_2

$A_1 + A_2 = 0$ and $-0.5A_1 - -2A_2 = 5$ to get $A_1 = 10/3$ and $A_2 = --10/3$

And the final solution for i(t) is : $(10/3)[e^{-0.5t} - e^{-2t}]$ Amps

(b) At time t=0 the voltage across the capacitor = 5V ie. $v_{C(t=0)}$ = 5V .
But i(t=0) is still =0.using these values in the equation (2) above we get
$(di/dt)_{@ t=0}$ = ½ (10-5) = 2-5
Then the two equations in A_1 and A_2 are $A_1 + A_2 = 0$ and $-0.5A_1 - -2A_2 = 2.5$
Solving these two equations we get $A_1 = 5/3$ and $A_2 = -5/3$

And the final solution for i(t) is : $(5/3)[e^{-0.5t} - e^{-2t}]$ Amps

(c) The roots of the characteristic equation when the Resistance is changed to

$4 s_1, s_2 = -R/2L \pm \sqrt{[(R/2L)^2 - (1/LC)]} = (-4/2x2) \pm \sqrt{[(4/2x2)^2 - (1/2x0.5)]} = -1.0$ i.e the roots are real and equal and the solution is given by

$$i(t) = e^{-\alpha t}(A_1 t + A_2) = e^{-1t}(A_1 t + A_2) \text{ --------- } (4)$$

Now using the initial condition i(t) = 0 at time t=0 we get $A_2 = 0$

We have already found in equation (3) for the basic series RLC circuit $(di/dt)_{@t=0} = 5$

Now we will find di(t)/dt of equation (4) and equate it to the above value.

di /dt = $-e^{-1t}(A_1t + A_2) + e^{-1t}(A_1) = e^{-1t}[A_1 - A_1t - A_2]$ and

(di /dt) $@t=0 = e^{-1x0}[A_1 - A_1x0 - A_2]$ i.e $A_1 - A_2 = 5$

Therefore A1 =5 and A2 = 0

And the final solution for i(t) is i(t) = $5te^{-1t}$ Amps

(d) Roots of the characteristic equation when the resistance is changed to 1 Ω are :

$s_1,s_2 = -R/2L \pm \sqrt{[(R/2L)^2 - (1/LC)]} = (-1/2x2) \pm \sqrt{[(1/4)^2 - (1/2x0.5)]} = --0.25 \pm j0.94$

The roots are complex and so the solution is then given by : **$i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$**

Where $\alpha = 0.25$ and $\omega_d = 0.9465$

Now we will apply the initial conditions to find out the constants A_1 and A_2

First initial condition is $i(t)@t=0 = 0$ applying this into the equation : $i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$ we get A1 = 0 and using this value of A1 in the above equation for i(t) we get

$i(t) = e^{-\alpha t}(A_2 \sin\omega_d t)$

We have already obtained the second initial condition as di (t) /dt$@t=0$= 5 from the basic equation of the series RLC circuit. Now let us differentiate above equation for current i.e :$i(t) = e^{-\alpha t}(A_2 \sin\omega_d t)$ and equate it to 5 to get the second constant A_2

di (t) /dt $= e^{-\alpha t}(A_2 \omega_d \cos\omega_d t) + (A_2 \sin\omega_d t) . -\alpha . e^{-\alpha t}$

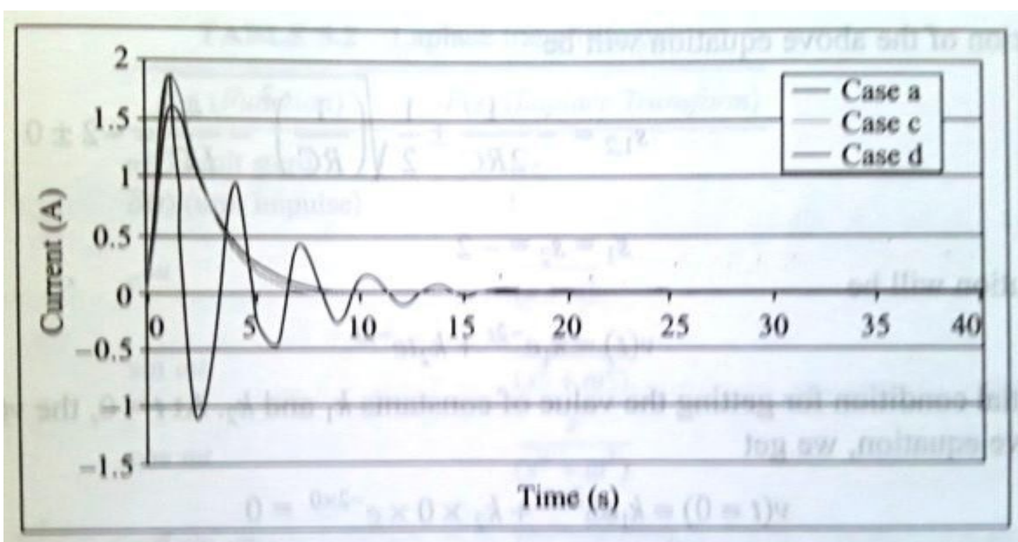
di (t) /dt $@t=0 = A_2 . \omega_d = 5$

i.e $A_2 = 5 / \omega_d = 5/0.94 = 5.3$

Now using this value of A_2 and the values of $\alpha = 0.25$ and $\omega_d = 0.94$in the above expression for the current we finally get :

$i(t) = e^{-0.25t}(2.569 \sin 1.9465t)$

The currents in all the three different cases (a), (c) and (d) are shown below :



UNIT-2

TWO PORT NETWORKS

- **Introduction**
- **Impedance Parameters**
- **Admittance Parameters**
- **Hybrid Parameters**
- **Transmission Parameters (ABC D)**
- **Conversion of one Parameter to other**
- **Conditions for reciprocity and symmetry**
- **Interconnection of two port networks in Series ,Parallel and Cascaded configurations**
- **Image parameters**
  - **Important Formulae, equations and relations Illustrative**
  - **problems**

Introduction:

A general network having two pairs of terminals, one labeled the "input terminals'' and the other the "output terminals,'' is a very important building block in electronic systems, communication systems, automatic control systems, transmission and distribution systems, or other systems in which an electrical signal or electric energy enters the input terminals, is acted upon by the network, and leaves via the output terminals. A pair of terminals at which a signal may enter or leave a network is also called a ***port,*** and a network like the above having two such pair of terminals is called a ***Two - port network.*** A general two-port network with terminal voltages and currents specified is shown in the figure below. In such networks the relation between the two voltages and the two currents can be described in six different ways resulting in six different systems of Parameters and in this chapter we will consider the most important four systems.

Impedance Parameters: Z parameters (open circuit impedance parameters)

We will assume that the two port networks that we will consider are composed of linear elements and contain no independent sources but dependent sources *are* permissible. We will consider the two-port network as shown in the figure below.

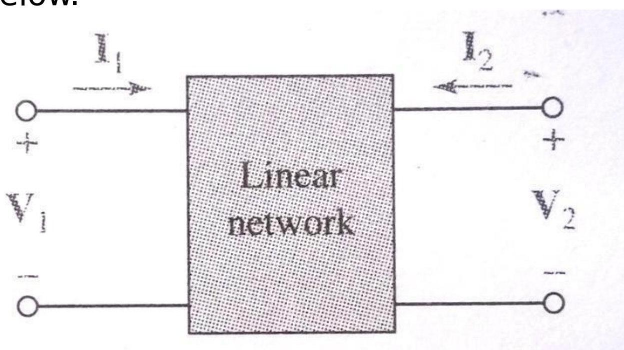


Fig: A general two-port network with terminal voltages and currents specified. The two-port network is composed of linear elements, possibly including dependent sources, but not containing any independent sources.

The voltage and current at the input terminals are $\mathbf{V}_1$ & $\mathbf{I}_1$, and $\mathbf{V}_2$ & $\mathbf{I}_2$ are voltage and current

at the output port. The directions of $\mathbf{I}_1$ and $\mathbf{I}_2$ are both customarily selected as *into* the network at the upper conductors (and out at the lower conductors). Since the network is linear and

contains no independent sources within it, $\mathbf{V}_1$ may be considered to be the superposition of two components, one caused by $\mathbf{I}_1$ and the other by $\mathbf{I}_2$. When the same argument is applied to $\mathbf{V}_2$, we get the set of equations

$$\mathbf{V}_1 = \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2$$

$$\mathbf{V}_2 = \mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2$$

This set of equations can be expressed in matrix notation as

$$\mathbf{V}_1 \quad \mathbf{Z}_{11} \quad \mathbf{Z}_{12} \quad \mathbf{I}_1$$

$$V_2 \quad Z_{21} \quad Z_{22} \quad I_2$$

And in much simpler form as

$$[V] = [Z][I]$$

Where **[V] ,[Z] and [I]** are Voltage, impedance and current matrices. The description of the **Z** parameters, defined in the above equations is obtained by setting each of the currents equal to zero as given below.

$$\mathbf{Z}_{11} = \mathbf{V}_1/\mathbf{I}_1 \Big|_{\mathbf{I}_2=0}$$
$$\mathbf{Z}_{12} = \mathbf{V}_1/\mathbf{I}_2 \Big|_{\mathbf{I}_1=0}$$
$$\mathbf{Z}_{21} = \mathbf{V}_2/\mathbf{I}_1 \Big|_{\mathbf{I}_2=0}$$
$$\mathbf{Z}_{22} = \mathbf{V}_2/\mathbf{I}_2 \Big|_{\mathbf{I}_1=0}$$

Thus ,Since zero current results from an open-circuit termination, the **Z** parameters are known as the ***Open-circuit Impedance parameters***. And more specifically $\mathbf{Z}_{11}$ & $\mathbf{Z}_{22}$ are called ***Driving point Impedances*** and $\mathbf{Z}_{12}$ & $\mathbf{Z}_{21}$ are called ***Reverse and Forward transfer impedances*** respectively.

A basic Z parameter equivalent circuit depicting the above defining equations is shown in the figure below.

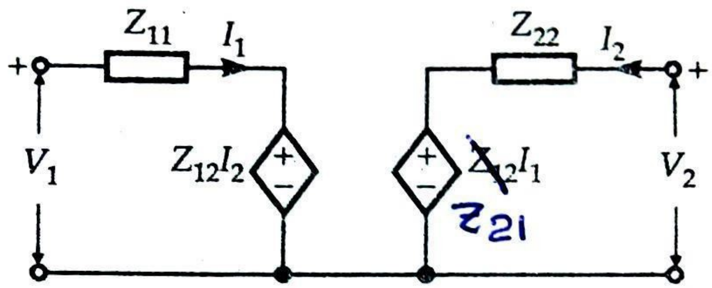


Fig: Z-Parameter equivalent circuit

Admittance parameters: (Y Parameters or Short circuit admittance parameters)

The same general two port network shown for **Z** parameters is applicable here also and is shown below.

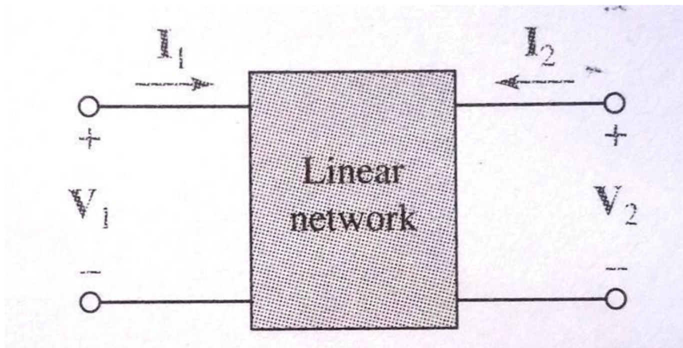


Fig: A general two-port network with terminal voltages and currents specified. The two-port network is composed of linear elements, possibly including dependent sources, but not containing any independent sources.

Since the network is linear and contains no independent sources within, on the same lines of **Z** parameters the defining equations for the Y parameters are given below. $\mathbf{I}_1$ and I_2 may be considered to be the superposition of two components, one caused by $\mathbf{V}_1$ and the other by $\mathbf{V}_2$ and then we get the set of equations defining the **Y** parameters.

$$\mathbf{I}_1 = \mathbf{Y}_{11}\mathbf{V}_1 + \mathbf{Y}_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = \mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2$$

where the **Y**s are no more than proportionality constants and their dimensions are A/V (Current/Voltage). Hence they are called the **Y** (or admittance) parameters. They are also defined in the matrix form given below.

$$\begin{matrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{matrix} \quad \begin{matrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{matrix} \quad \begin{matrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{matrix}$$

And in much simpler form as

$$[\mathbf{I}] = [\mathbf{Y}][\mathbf{V}]$$

The individual Y parameters are defined on the same lines as Z parameters but by setting either of the voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ as zero as given below.

The most informative way to attach a physical meaning to the **y** parameters is through a direct inspection of defining equations. The conditions which must be applied to the basic defining equations are very important. In the first equation for example; if we let $\mathbf{V}_2$ zero, then $\mathbf{Y}_{11}$ is given by the ratio of $\mathbf{I}_1$ to $\mathbf{V}_1$. We therefore describe $\mathbf{Y}_{11}$ as the admittance measured at the input terminals with the output terminals *short-circuited* ($\mathbf{V}_2 = 0$). Each of the **Y** parameters may be described as a

current-voltage ratio with either **V**1 = 0 (the input terminals short circuited) or **V**2 = 0 (the output terminals short-circuited):

$Y_{11} = I_1/V_1$ with $V_2 = 0$
$Y_{12} = I_1/V_2$ with $V_1 = 0$
$Y_{21} = I_2/V_1$ with $V_2 = 0$
$y_{22} = I_2/V_2$ with $V_1 = 0$

Because each parameter is an ***admittance*** which is obtained by short circuiting either the output or the input port, the **Y** parameters are known as the ***short-circuit admittance parameters.*** The specific name of Y_{11} is the ***short-circuit input admittance,*** Y_{22} is the ***short-circuit output admittance,*** and Y_{12} and Y_{21} are the ***short-circuit reverse and forward transfer admittances respectively.***

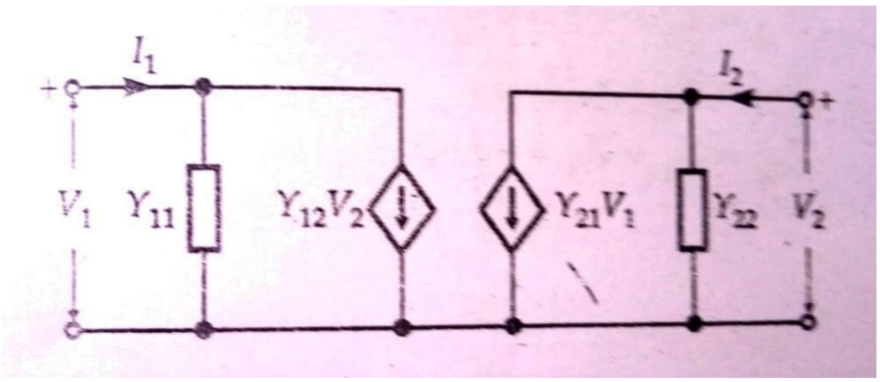


Fig: Y parameter equivalent circuit

Hybrid parameters: (h parameters)

h parameter representation is used widely in modeling of Electronic components and circuits particularly Transistors. Here both short circuit and open circuit conditions are utilized.
The hybrid parameters are defined by writing the pair of equations relating V_1, I_1, V_2, and I_2:

$$V_1 = h_{11}.I_1 + h_{12}.V_2$$
$$I_2 = h_{21}.I_1 + h_{22}.V_2$$

Or in matrix form :

$$\begin{matrix} V_1 \\ I_2 \end{matrix} \quad h \quad \begin{matrix} I_1 \\ V_2 \end{matrix}$$

The nature of the parameters is made clear by first setting $V_2 = 0$. Thus,

$h_{11} = V_1/I_1$ with $V_2 = 0$ = ***short-circuit input impedance***

$h_{21} = I_2/I_1$ with $V_2 = 0$ = ***short-circuit forward current gain***

Then letting $I_1 = 0$, we obtain

$h_{12} = V_1/V_2$ **with** $I_1=0$ ***= open-circuit reverse voltage gain***

$h_{22} = I_2/V_2$ **with** $I_1=0$ ***= open-circuit output admittance***

Since the parameters represent an impedance, an admittance, a voltage gain, and a current gain, they are called the "hybrid'' parameters.

The subscript designations for these parameters are often simplified when they are applied to transistors. Thus, h_{11}, h_{12}, h_{21}, and h_{22} become h_i, h_r, h_f, and h_o, respectively, where the subscripts denote input, reverse, forward, and output.

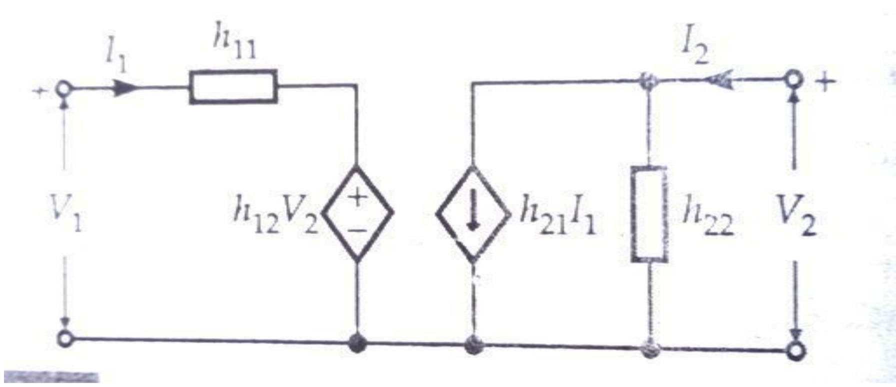


Fig: h parameter equivalent circuit

Transmission parameters:

The last two-port parameters that we will consider are called the ***t parameters,*** the ***ABCD parameters,*** or simply the ***transmission parameters.*** They are defined by the equations

$$V_1 = A.V_2 - B.I_2$$
$$I_1 = C.V_2 - D.I_2$$

and in Matrix notation these equations can be written in the form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where V_1, V_2, I_1, and I_2 are defined as as shown in the figure below.

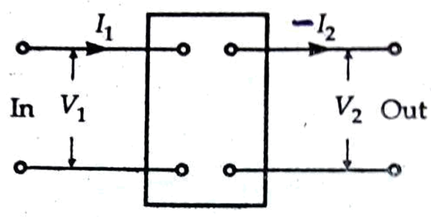


Fig: Two port Network for ABCD parameter representation with Input and output Voltages and currents

The minus signs that appear in the above equations should be associated with the output current, as ($-\mathbf{I}_2$). Thus, both $\mathbf{I}_1$ and $-\mathbf{I}_2$ are directed to the right, the direction of energy or signal transmission.

Note that there are no minus signs in the **t** or **ABCD** matrices. Looking again at the above equations we see that the quantities on the left, often thought of as the given or independent

variables, are the input voltage and current, $\mathbf{V}_1$ and $\mathbf{I}_1$; the dependent variables, $\mathbf{V}_2$ and $\mathbf{I}_2$, are the output quantities. Thus, the transmission parameters provide a direct relationship between input and output. Their major use arises in transmission-line analysis and in cascaded networks.

The four Transmission parameters are defined and explained below.

*First **A** and **C** are defined with receiving end open circuited i.e. with $I_2 = 0$*

A =	**with**		**= Reverse**
V_1/V_2	**I_2**	**= 0**	**voltage Ratio**
C =	**with**		**= Transfer**
I_1/V_2	**I_2**	**= 0**	**admittance**

*Next **B** and **D** are defined with receiving end short circuited i.e. with $V_2 = 0$*

B =	**with V_2 =**	**= Transfer**
$V_1/-I_2$	**0**	**impedance**
D = I /−I	**with V_2 =**	**= Reverse**
1 2	**0**	**current ratio**

Inter relationships between different parameters of two port networks:

Basic Procedure for representing any of the above four two port Network parameters in terms of the other parameters consists of the following steps:

1. *Write down the defining equations corresponding to the parameters in terms of which the other parameters are to be represented.*
2. *Keeping the basic parameters same, rewrite/manipulate these two equations in such a*

   way that the variables V_1 ,V_2 ,I_1 ,and I_2 are arranged corresponding to the defining equations of the first parameters.

3. *Then by comparing the parameter coefficients of the respective variables V_1 ,V_2 ,I_1 ,and I_2 on the right hand side of the two sets of equations we can get the inter relationship.*

Z Parameters in terms of Y parameters:

Though this relationship can be obtained by the above steps, the following simpler method is used for Z in terms of Y and Y in terms of Z:

Z and Y being the Impedance and admittance parameters (Inverse), in matrix notation they are governed by the following inverse relationship.

$$[Z] = [Y]^{-1}$$

Or:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1}$$

Thus :

$$Z_{11} = \frac{Y_{22}}{\Delta Y}, Z_{12} = -\frac{Y_{12}}{\Delta Y}$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} \text{ and } Z_{22} = \frac{Y_{11}}{\Delta Y}$$

$$\left[\text{Here } \Delta Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = Y_{11}\,Y_{22} - Y_{12}\,Y_{21}\right]$$

Z Parameters in terms of ABCD parameters:

The governing equations are:

$V_1 = AV_2 - BI_2$

$I_1 = CV_2 - DI_2$

from the second governing equation [$I_1 = CV_2 - DI_2$] we can write

$$V_2 = \frac{1}{C} \cdot I_1 + \frac{D}{C} \cdot I_2$$

Now substituting this value of V_2 in the first governing equation [$V_1 = AV_2 - BI_2$] we get

$$V_1 = \left[\frac{1}{C} \cdot I_1 + \frac{D}{C} \cdot I_2\right] \; A - BI_2$$

$$= \frac{A}{C} \cdot I_1 + \frac{AD - BC}{C} \cdot I_2$$

Comparing these two equations for **V_1** and **V_2** with the governing equations of the **Z** parameter network we get **Z** Parameters in terms of ABCD parameters:

$$Z_{11} = \frac{A}{C}, Z_{12} = \frac{AD - BC}{C}$$

$$Z_{21} = \frac{1}{C}, Z_{22} = \frac{D}{C}$$

Z Parameters in terms of h parameters:

The governing equations of h parameter network are:

$V_1 = h_{11} I_1 + h_{12} V_2$
$I_2 = h_{21} I_1 + h_{22} V_2$

From the second equation we get

$$V_2 = -\frac{h_{21}}{h_{22}} \cdot I_1 + \frac{1}{h_{22}} \cdot I_2$$

Substituting this value of V_2 in the first equation for V_1 we get:

$$\begin{aligned} V_1 &= h_{11} I_1 + h_{12} V_2 \\ &= h_{11} I_1 + h_{12} \left[-\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} \cdot I_2 \right] \\ &= \frac{\Delta h}{h_{22}} I_1 + \frac{h_{12}}{h_{22}} \cdot I_2 \end{aligned}$$

Now comparing these two equations for V_1 and V_2 with the governing equations of the **Z** parameter network we get **Z** Parameters in terms of **h** parameters:

$$Z_{11} = \frac{\Delta h}{h_{22}}, \quad Z_{12} = \frac{h_{12}}{h_{22}}$$
$$Z_{21} = -\frac{h_{21}}{h_{22}}, Z_{22} = \frac{1}{h_{22}}$$

Here h = $h_{11} h_{22} - h_{12} h_{21}$

Y Parameters in terms of Z parameters:

Y and Z being the admittance and Impedance parameters (Inverse), in matrix notation they are governed by the following inverse relationship.

Or:

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1}$$

Thus:

$$Y_{11} = \frac{Z_{22}}{\Delta Z}, \quad Y_{12} = -\frac{Z_{12}}{\Delta Z}$$
$$Y_{21} = -\frac{Z_{21}}{\Delta Z}, Y_{22} = \frac{Z_{11}}{\Delta Z}$$

Here $\Delta Z = z_{11} z_{22} - z_{12} z_{21}$

The other inter relationships also can be obtained on the same lines following the ***basic three steps given*** in the beginning.

Conditions for reciprocity and symmetry in two port networks:

A two port network is said to be ***reciprocal*** if the ratio of the output response variable to the input excitation variable is same when the excitation and response ports are interchanged.

A two port network is said to be ***symmetrical*** if the port voltages and currents remain the same when the input and output ports are interchanged.
In this topic we will get the conditions for ***Reciprocity*** and ***symmetry*** for all the four networks.
The basic procedure for each of the networks consists of the following steps:
Reciprocity:

- First we will get an expression for the ratio of response to the excitation in terms of the **particular parameters** by giving voltage as excitation at the input port and considering the current in the output port as response (by short circuiting the output port i.e setting V_2 as zero). i.e find out (I_2 / V_1)
- Then we will get an expression for the ratio of response to the excitation in terms of the **same parameters** by giving voltage as excitation at the output port and considering the current in the input port as response (by short circuiting the input port i.e. setting V_1 as zero). i.e find out (I_1 / V_2)
- Equating the RHS of these two expressions would be the condition for reciprocity

Symmetry:

- First we need to get expressions related to the input and output ports using the basic Z or Y parameter equations.
- Then the expressions for Z_{11} and Z_{22} (or Y_{11} and Y_{22}) are equated to get the conmdition for reciprocity.

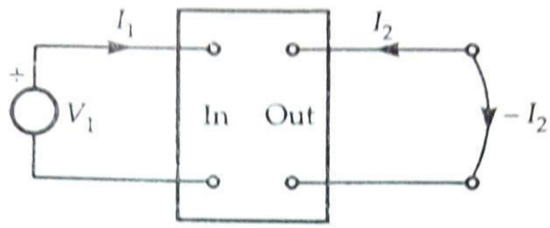
Z parameter representation:

Condition for reciprocity:

Let us take a two port network with **Z** parameter defining equations as given below:

$$\mathbf{V}_1 = \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2$$
$$\mathbf{V}2 = \mathbf{Z}21\mathbf{I}1 + \mathbf{Z}22\mathbf{I}2$$

First we will get an expression for the ratio of response (I_2) to the excitation (V_1) in terms of the **Z parameters** by giving excitation at the input port and considering the current in the output port as response (by short circuiting the output port i.e. setting V_2 as zero).The corresponding **Z** parameter circuit for this condition is shown in the figure below:



(Pl note the direction of I_2 is negative since when V_2 port is shorted the current flows in the other direction)

Then the Z parameter defining equations are :

$V_1 = Z_{11} . I_1 - Z_{12}. I_2$ and

$0 = Z_{21} . I_1 - Z_{22}. I_2$

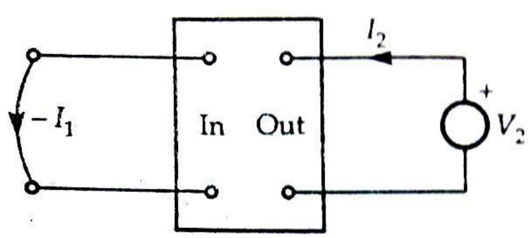
*To get the ratio of response **(I_2)** to the excitation **(V_1)** in terms of the Z parameters **I_1** is to be eliminated fom the above equations.*

So from equation 2 in the above set we will get **$I_1 = I_2. Z_{22}/ Z_{21}$** And substitute this in the first equation to get

$V_1 = (Z_{11} .I_2. Z_{22}/ Z_{21}) - Z_{12}. I_2 = I_2 [(Z_{11} . Z_{22}/ Z_{21}) - Z_{12}] = I_2 [(Z_{11} . Z_{22} - Z_{12}.Z_{21}) / Z_{21})]$

$I_2 = V_1 . Z_{21}/(Z_{11} . Z_{22} - Z_{12}.Z_{21})$

Next, we will get an expression for the ratio of response (**I_1**) to the excitation **(V_2)** in terms of the **Z parameters** by giving excitation **V_2** at the output port and considering the current **I_1** in the input port as response (by short circuiting the input port i.e. setting **V_1** as zero). The corresponding **Z** parameter circuit for this condition is shown in the figure below:



(Pl note the direction of current I_1 is negative since when V_1 port is shorted the current flows in the other direction)

Then the Z parameter defining equations are :

$\mathbf{0 = -Z_{11} . I_1 + Z_{12}. I_2}$ and

$\mathbf{V_2 = -Z_{21} . I_1 + Z_{22}. I_2}$

*To get the ratio of response (**I_1**) to the excitation **(V_2)** in terms of the Z parameters **I_2** is to be eliminated fom the above equations.*

So from equation 1 in the above set we will get **$I_2 = I_1 . Z_{11} / Z_{12}$** And substitute this in the second equation to get

$V_2 = (Z_{22}.I_1. Z_{11}/ Z_{12}) - Z_{21}. I_1 = I_1 [(Z_{11} . Z_{22}/ Z_{12}) - Z_{21}] = I_1 [(Z_{11} . Z_{22} - Z_{12}.Z_{21}) / Z_{12})]$

$I_1 = V_2 . Z_{12}/(Z_{11} . Z_{22} - Z_{12}.Z_{21})$

Assuming the input excitations **V_1** and **V_2** to be the same, then the condition for the out responses **I_1** and **I_2** to be equal would be

$$Z_{12} = Z_{21}$$

And this is the condition for the reciprocity.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports using the basic Z parameter equations.

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

To get the input port impedance I_2 is to be made zero. i.e V_2 should be open.

$$V_1 = Z_{11} . I_1 \text{ i.e } Z_{11} = V_1/I_1 \mid I_2=0$$

Similarly to get the output port impedance I_1 is to be made zero. i.e V_1 should be open.

$$V_2 = Z_{22} . I_2 \text{ i.e } Z_{22} = V_2/I_2 \mid I_1=0$$

Condition for Symmetry is obtained when the two port voltages are equal i.e. $V_1 = V_2$ and the two port currents are equal i.e. $I_1 = I_2$. Then

$V_1/I_1 = V_2/I_2$ i.e $Z_{11} = Z_{22}$

And hence $Z_{11} = Z_{22}$ is the condition for symmetry in Z parameters .

Y parameter representation:

Condition for reciprocity :

Let us take a two port network with **Y** parameter defining equations as given below:

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$\mathbf{I}_2 = \mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2$$

First we will get an expression for the ratio of response **(I_2)** to the excitation **(V_1)** in terms of the **Y parameters** by giving excitation **(V_1)** at the input port and considering the current **(I_2)** in the output port as response (by short circuiting the output port i.e. setting **V_2** as zero) Then the second equation in **Y** parameter defining equations would become

$$I_2 = Y_{21}V_1 + 0 \text{ and } I_2 / V_1 = Y_{21}$$

Then we will get an expression for the ratio of response **(I_1)** to the excitation **(V_2)** in terms of the **Y parameters** by giving excitation **(V_2)** at the output port and considering the current **(I_1)** in the input port as response (by short circuiting the input port i.e setting **V_1** as zero) Then the first equation in Y parameter defining equations would become

$$I_1 = 0 + Y_{12}V_2 \text{ and } I_1 / V_2 = Y_{12}$$

Assuming the input excitations **V_1** and **V_2** to be the same, then the condition for the out responses **I_1** and **I_2** to be equal would be

$$I_1 / V_2 = I_2 / V_1$$

And hence $Y_{12} = Y_{21}$ is the condition for the reciprocity in the Two port network with Y parameter representation.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports (In this case Input and output admittances) using the basic Y parameter equations

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

To get the input port admittance, **V_2** is to be made zero. i.e **V_2** should be shorted.

$$I_1 = Y_{11} \cdot V_1 \text{ i.e } Y_{11} = I_1/V_1 \Big| V_2=0$$

Similarly to get the output port admittance V_1 is to be made zero. i.e V_1 should be shorted.

$$I_2 = Y_{22} \cdot V_2 \text{ i.e } Y_{22} = I_2/V_2 \Big| V_1=0$$

Condition for Symmetry is obtained when the two port voltages are equal i.e. **$V_1 = V_2$** and the two port currents are equal i.e**. $I_1 = I_2$.** Then

$I_1/V_1 = I_2/V_2$

And hence $Y_{11} = Y_{22}$ is the condition for symmetry in Y parameters.

ABCD parameter representation:

Condition for reciprocity :

Let us take a two port network with ABCD parameter defining equations as given below:

$$V_1 = A.V_2 - B.I_2$$
$$I_1 = C.V_2 - D.I_2$$

First we will get an expression for the ratio of response **(I_2)** to the excitation **(V_1)** in terms of the **ABCD parameters** by giving excitation **(V_1)** at the input port and considering the current **(I_2)** in the output port as response (by short circuiting the output port i.e. setting **V_2** as zero) Then the first equation in the **ABCD** parameter defining equations would become

$$V_1 = 0 - B.I_2 = B.I_2$$
$$\text{i.e } I_2 / V_1 = -1/B$$

Then we will interchange the excitation and response i.e. we will get an expression for the ratio of response **(I_1)** to the excitation **(V_2)** by giving excitation **(V_2)** at the output port and considering the current **(I_1)** in the input port as response (by short circuiting the input port i.e. setting **V_1** as zero)
Then the above defining equations would become

$$0 = A.V_2 - B.I_2 \qquad I_1 = C.V_2 - D.I_2$$

Substituting the value of **$I_2 = A.V_2 /B$** from first equation into the second equation we get

$$I_1 = C.V_2 - D.\ A.V_2 /B = V_2 (C - D.\ A /B)$$
$$\text{i.e } I_1/V_2 = (BC - DA) / B = -(AD - BC)/B$$

Assuming the input excitations **V_1** and **V_2** to be the same , then the condition for the out responses **I_1** and **I_2** to be equal would be

$$I_1 / V_2 = I_2 / V_1$$
$$\text{i.e } -(AD - BC)/B = -1/B$$
$$\text{i.e } (AD - BC) = 1$$

And hence AD – BC = 1 is the condition for Reciprocity in the Two port network with ABCD parameter representation.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports. In this case it is easy to use the Z parameter definitions of Z_{11} and Z_{22} for the input and output ports respectively and get their values in terms of the ABCD parameters as shown below.

$$V_1 = A.V_2 - B.I_2$$
$$I_1 = C.V_2 - D.I_2$$

$$\mathbf{Z}_{11} = \mathbf{V}_1/\mathbf{I}_1 \Big|_{\mathbf{I}_2=0}$$

Applying this in both the equations we get

$$\mathbf{Z}_{11} = \mathbf{V}_1/\mathbf{I}_1 \Big|_{\mathbf{I}_2=0} = (\mathbf{A.V_2 - B.I_2})/(\mathbf{C.V_2 - D.I_2}) \Big|_{\mathbf{I}_2=0}$$

$$= (\mathbf{A.V_2 - B.0})/(\mathbf{C.V_2 - D.0})$$

$$= (\mathbf{A.V_2})/(\mathbf{C.V_2}) = \mathbf{A/C}$$

$$\mathbf{Z_{11} = A/C}$$

Similarly $Z_{22} = V_2/I_2 \,\big|\, I_1=0$

and using this in the second basic equation $I_1 = C.V_2 - D.I_2$

we get $0 = C.V_2 - D.I_2$ or $C.V_2 = D.I_2$

$$V_2 / I_2 = D/C$$

$Z_{22} = D/C$

And the condition for symmetry i.e $Z_{11} = Z_{22}$ becomes $A/C = D/C$

Or $A = D$

Hence A = D is the condition for Symmetry in ABCD parameter representation.

h parameter representation:

Condition for reciprocity :

Let us take a two port network with h parameter defining equations as given below:

$$V_1 = h_{11}.I_1 + h_{12}.V_2$$

$$I_2 = h_{21}.I_1 + h_{22}.V_2$$

First we will get an expression for the ratio of response **(I_2)** to the excitation **(V_1)** in terms of the **h parameters** by giving excitation **(V_1)** at the input port and considering the current **(I_2)** in the output port as response (by short circuiting the output port i.e. setting V_2 as zero)

Then the first equation in the **h** parameter defining equations would become

$$V_1 = h_{11}.I_1 + h_{12}.0 = h_{11}.I_1$$

And in the same condition the second equation in the **h** parameter defining equations would become

$$I_2 = h_{21}.I_1 + h_{22}.0 = h_{21}.I_1$$

Dividing the second equation by the first equation we get

$$I_2 / V_1 = (h_{21}.I_1) / (h_{11}.I_1) = h_{21} / h_{11}$$

Now the excitation and the response ports are interchanged and then we will get an expression for the ratio of response **(I_1)** to the excitation **(V_2)** in terms of the **h parameters** by giving excitation **(V_2)** at the output port and considering the current **(I_1)** in the input port as response (by short circuiting the input port i.e. setting V_1 as zero)

Then the first equation in **h** parameter defining equations would become

$$0 = h_{11}.I_1 + \quad h_{11}.I_1 = -$$

$\mathbf{h_{12}.V_2}$ **i.e** $\mathbf{h_{12}.V_2}$

i.e. $\mathbf{I}_1$ /

$\mathbf{V}_2$ = $\mathbf{- h_{12} / h_{11}}$

Assuming the input excitations $\mathbf{V_1}$ and $\mathbf{V_2}$ to be the same, then the condition for the out responses $\mathbf{I_1}$ and $\mathbf{I_2}$ to be equal would be

$\mathbf{I}_1 / \mathbf{V}_2 = \mathbf{I}_2 / \mathbf{V}_1$ **i.e.** $\mathbf{h_{12} = - h_{21}}$

i.e = $\mathbf{- h_{12} / h_{11} = h_{21} / h_{11}}$

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports. In this case also it is easy to use the Z parameter definitions of $\mathbf{Z_{11}}$ and $\mathbf{Z_{22}}$ for the input and output ports respectively and get their values in terms of the **h** parameters as shown below.

h parameter equations are : $\mathbf{V_1 = h_{11}.\ I_1 + h_{12}.V_2}$

$$\mathbf{I_2 = h_{21}.\ I_1 + h_{22}.V_2}$$

First let us get Z_{11} :

$$\mathbf{Z}_{11} = \mathbf{V}_1/\mathbf{I}_1 \Big|\ \mathbf{I}_2=0$$

$$= \mathbf{h_{11} + h_{12}.V_2 / I_1}$$

Applying the condition $\mathbf{I}_2=0$ in the equation 2 we get

$$\mathbf{0 = h_{21}.\ I_1 + h_{22}.V_2 \text{ i.e } -h_{21}.\ I_1 = h_{22}.V_2}$$

$$\mathbf{\text{or } V_2 = I_1\ (-h_{21} / h_{22})}$$

Now substituting the value of $\mathbf{V_2 = I_1\ (-h_{21} / h_{22})}$ in the above first expression for $\mathbf{V_1}$ we get

$$\mathbf{V_1} = \mathbf{h_{11}.\ I_1 + h_{12}.\ I_1.(\ -h_{21} / h_{22}\ )}$$

$$\text{Or } \mathbf{V_1/I_1} = (\mathbf{h_{11}.\ h_{22} - h_{12}.\ h_{21}\ )/\ h_{22} = \Delta h / h_{22}}$$

$$\mathbf{Or\ \ Z}_{11} = \Delta h\ \mathbf{/ h_{22}}$$

Where **h = (h11. h22 – h12. h21)**

Now let us get Z_{22} :

$$\mathbf{Z_{22}} = \mathbf{V}_2/\mathbf{I_2}\Big|\ \mathbf{I}_1 = 0$$

Applying the condition $\mathbf{I}_1 = 0$ in the second equation we get

$$\mathbf{I_2 = h_{21}.\ 0 + h_{22}.V_2} \quad \mathbf{\text{i.e } V_2/I_2 = 1/\ h_{22}}$$

$$\mathbf{\text{And } Z_{22} = 1/\ h_{22}}$$

Hence the condition for symmetry $\mathbf{Z}_{11} = \mathbf{Z_{22}}$ becomes **(h / h22)** = **(1/ h22) i.e** **h = 1**

Hence h = 1 is the condition for symmetry in h parameter representation.

Parameter	Condition for reciprocity	Condition for symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{21}$
h	$h_{12} = -h_{21}$	$\Delta h = 1$
$ABCD$	$AD - BC = 1$	$A = D$

Different types of interconnections of two port networks:

Series Connection:

Though here only two networks are considered, the result can be generalized for any number of two port networks connected in series.

Refer the figure below where two numbers of two port networks **A** and **B** are shown connected in series. All the input and output currents & voltages with directions and polarities are shown.

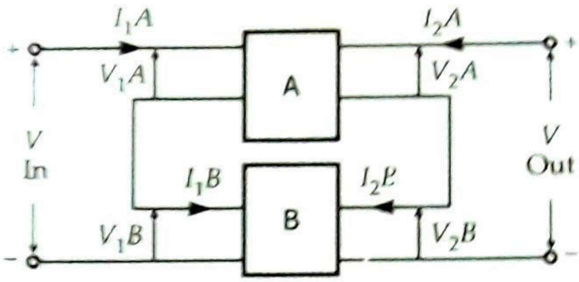


Fig : Series connection of two numbers of Two Port Networks

***Open circuit Impedance parameters* (Z)** are used in characterizing the Series connected Two port Networks .The governing equations with **Z** parameters are given below:

For network A :

$$V_{1A} = Z_{11A}\ I_{1A} + Z_{12A}\ I_{2A}$$

$$V_{2A} = Z_{21A}\ I_{1A} + Z_{22A}\ I_{2A}$$

And for network B:

$$V_{1B} = Z_{11B}\ I_{1B} + Z_{12B}\ I_{2B}$$

$$V_{2B} = Z_{21B}\ I_{1B} + Z_{22B}\ I_{2B}$$

Referring to the figure above the various voltage and current relations are:

$$I_1 \equiv I_{1A} \equiv I_{1B}$$

$$I_2 \equiv I_{2A} \equiv I_{2B}$$

$$V_2 = V_{2A} + V_{2B}$$

$$V_1 = V_{1A} + V_{1B}$$

Now substituting the above basic defining equations for the two networks into the above expressions for **V1** and **V2** and using the above current equalities we get:

$$\begin{aligned} V_1 &= V_{1A} + V_{1B} \\ &= (Z_{11A} I_{1A} + Z_{12A} I_{2A}) + Z_{11B} I_{1B} + Z_{12B} I_{2B} \\ &= I_1\,(Z_{11A} + Z_{11B}) + I_2\,(Z_{12A} + Z_{12B}) \end{aligned}$$

And similarly

$$\begin{aligned} V_2 &= V_{2A} + V_{2B} \\ &= (Z_{21A} I_{1A} + Z_{22A} I_{2A}) + (Z_{21B} I_{1B} + Z_{22B} I_{2B}) \end{aligned}$$

$$V_2 = I_1\,(Z_{21A} + Z_{21B}) + I_2\,(Z_{22A} + Z_{22B})$$

Thus we get for two numbers of series connected two port networks:

$$V_1 = (Z_{11A} + Z_{11B})\, I_1 + (Z_{12A} + Z_{12B})\, I_2$$

$$V_2 = (Z_{21A} + Z_{21B})\, I_1 + (Z_{22A} + Z_{22B})\, I_2$$

Or in matrix form:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11A} + Z_{11B} & Z_{12A} + Z_{12B} \\ Z_{21A} + Z_{21B} & Z_{22A} + Z_{22B} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus it can be seen that the Z parameters for the series connected two port networks are the sum of the Z parameters of the individual two port networks.

Cascade connection:

In this case also though here only two networks are considered, the result can be generalized for any number of two port networks connected in cascade.

Refer the figure below where two numbers of two port networks **X** and **Y** are shown connected in cascade. All the input and output currents & voltages with directions and polarities are shown.

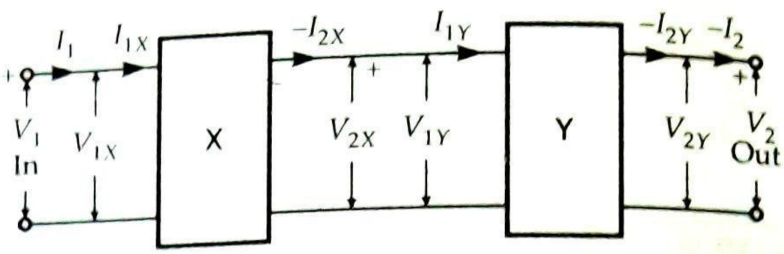


Fig: Two numbers of two port networks connected in cascade

Transmission (ABCD) parameters are easily used in characterizing the cascade connected
Two port Networks .The governing equations with transmission parameters are given below:

For network X:

$$V_{1X} = A_X\, V_{2X} - B_X I_{2X}$$
$$I_{IX} = C_X\, V_{2X} - D_X\, I_{2X}$$

And for network Y:

$$V_{1Y} = A_Y\, V_{2Y} - B_Y I_{2Y}$$
$$I_{1Y} = C_Y V_{2Y} - D_Y I_{2Y}$$

Referring to the figure above the various voltage and current relations are:

$$I_1 = I_{1X}\ ; -I_{2X} = I_{1Y}\ ; I_2 = I_{2Y}$$
$$V_1 = V_{IX}\ ; V_{2X} = V_{1Y}\ ; V_2 = V_{2Y}$$

Then the overall transmission parameters for the cascaded network in matrix form will become

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{1X} \\ I_{1X} \end{bmatrix} = \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} V_{2X} \\ -I_{2X} \end{bmatrix}$$

$$= \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} V_{1Y} \\ I_{1Y} \end{bmatrix}$$

$$= \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ C_Y & D_Y \end{bmatrix} \begin{bmatrix} V_{2Y} \\ -I_{2Y} \end{bmatrix}$$

$$= \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ C_Y & D_Y \end{bmatrix} \begin{bmatrix} V_Y \\ -I_Y \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_Y \\ -I_Y \end{bmatrix}$$

Where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ D_Y & D_Y \end{bmatrix}$$

Thus it can be seen that the overall ABCD Parameter matrix of cascaded two Port Networks is the product of the ABCD matrices of the individual networks.

Parallel Connection:

Though here only two networks are considered, the result can be generalized for any number of two port networks connected in parallel.

Refer the figure below where two numbers of two port networks **A** and **B** are shown connected in parallel. All the input and output currents & voltages with directions and polarities are shown.

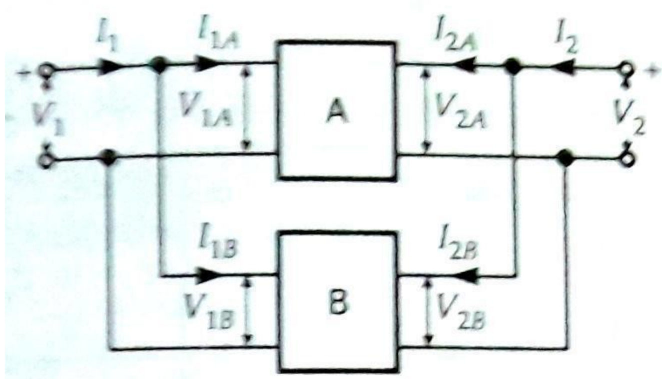


Fig: Parallel connection of two numbers of Two Port Networks

Short circuit admittance (Y) parameters are easily used in characterizing the parallel connected
Two port Networks .The governing equations with Y parameters are given below:

For network A:

$$I_{1A} = Y_{11A}\ V_{1A} + Y_{12A}\ V_{2A}$$
$$I_{2A} = Y_{21A}\ V_{1A} + Y_{22A}\ V_{2A}$$

And for network B:

$$I_{1B} = Y_{11B}\ V_{1B} + Y_{12B}\ V_{2B}$$
$$I_{2B} = Y_{21B}\ V_{1B} + Y_{22B}\ V_{2B}$$

Referring to the figure above the various voltage andcurrent relations are:

$$V_1 = V_{1A} = V_{1B} ; V_2 = V_{2A} = V_{2B}$$

$$I_1 = I_{1A} + I_{1B} ; I_2 = I_{2A} + I_{2B}$$

Thus

$$\begin{aligned} I_1 &= I_{1A} + I_{1B} \\ &= (Y_{11A}V_{1A} + Y_{12A}V_{2A}) + (Y_{11B}V_{1B} + Y_{12B}V_{2B}) \\ &= (Y_{11A} + Y_{11B})\,V_1 + (Y_{12A} + Y_{12B})\,V_2 \end{aligned}$$

$$\begin{aligned} I_2 &= I_{2A} + I_{2B} \\ &= (Y_{21A}V_{1A} + Y_{22A}V_{1B}) + (Y_{21B}V_{1B} + Y_{22B}V_{2B}) \\ &= (Y_{21A} + Y_{21B})\,V_1 + (Y_{22A} + Y_{22B})\,V_2 \end{aligned}$$

Thus we finally obtain the Y parameter equations for the combined network as:

$$I_1 = (Y_{11A} + Y_{11B})V_1 + (Y_{12A} + Y_{12B})V_2$$
$$I_2 = (Y_{21A} + Y_{21B})V_1 + (Y_{22A} + Y_{22B})V_2$$

And in matrix notation it will be:

$$\begin{bmatrix} I_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} Y_{11A} + Y_{11B} & Y_{12A} + Y_{12B} \\ Y_{21A} + Y_{21B} & Y_{22A} + Y_{22B} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus it can be seen that the overall Y parameters for the parallel connected two port networks are the sum of the Y parameters of the individual two port networks.

Image impedances in terms of ABCD parameters:

Image impedances Zi1 and Zi2 of a two port network as shown in the figure below are defined as two values of impedances such that :

a) When port two is terminated with an impedance **Zi2** , the input impedance as seen from Port one is **Zi1** and

b) When port one is terminated with an impedance **Zi1** , the input impedance as seen from Port two is **Zi2**

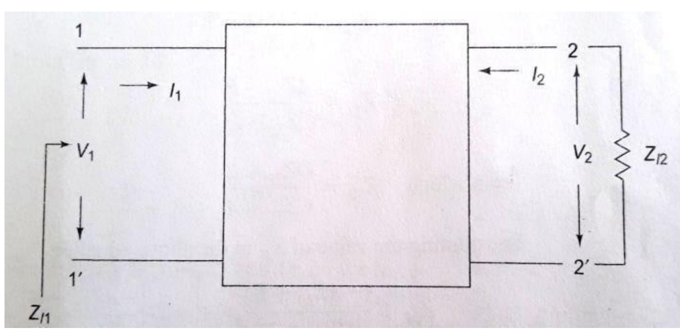


Figure pertining to condition (a) above

Corresponding Relations are : $Z_{i1} = V_1 / I_1$ and $Z_{i2} = V_2 / -I_2$

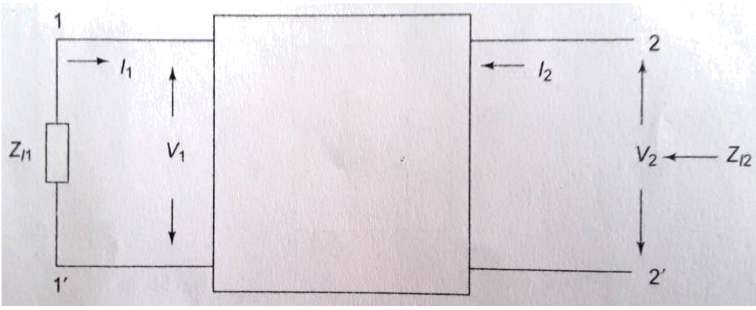


Figure pertining to condition (b) above

Corresponding Relations are : $Z_{i1} = V_1 / – I_1$ and $Z_{i2} = V_2 / I_2$

Such Image impedances in terms of ABCD parameters for a two port network are obtained below:
The basic defining equations for a two port network with ABCD parameters are :

$$\mathbf{V_1 = A.V_2 – B.I_2}$$
$$\mathbf{I_1 = C.V_2 – D.I_2}$$

First let us consider condition (a).
Dividing the first equation with the second equation we get

$$Z_{i1} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

But we also have $\mathbf{Z_{i2} = V_2 / – I_2}$ and so $\mathbf{V_2 = – Z_{i2}\, I_2}$. Substituting this value of V_2 in the above we get

$$Z_{i1} = \frac{-AZ_{i2} - B}{-CZ_{i2} - D} = \frac{AZ_{i2} + B}{CZ_{i2} + D}$$

Now let us consider the condition (b):
The basic governing equations [$\mathbf{V_1 = A.V_2 – B.I_2}$ **] and [**$\mathbf{I_1 = C.V_2 – D.I_2}$ **]** are manipulated to get

$$V_2 = \frac{DV_1}{AD - BC} - \frac{BI_1}{AD - BC}$$

$$I_2 = \frac{CV_1}{AD - BC} - \frac{AI_1}{AD - BC}$$

$$Z_{i2} = \frac{V_2}{I_2} = \frac{DV_1 - BI_1}{CV_1 - AI_1}$$

But we also have $\mathbf{Z_{i1} = V_1 / – I_1}$ and so $\mathbf{V_1 = – Z_{i1}\, I_1}$. Substituting this value of $\mathbf{V_1}$ in the above we get :

$$Z_{i2} = \frac{DZ_{i1} + B}{CZ_{i1} + A}$$

Z -

Y -

h
Parameters :

Solving the above equations for Z_{i1} and Z_{i2} we get :

$$Z_{i1} = \sqrt{\frac{AB}{CD}}\ ; \quad Z_{i2} = \sqrt{\frac{BD}{AC}}$$

Important formulae, Equations and Relations:

- **Basic Governing equations in terms of the various Parameters:**

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

$$V_1 = h_{11}.I_1 + h_{12}.V_2$$

$$I_2 = h_{21}.I_1 + h_{22}.V_2$$

$$V_1 = A.V_2 - B.I_2$$

$$I_1 = C.V_2 - D.I_2$$

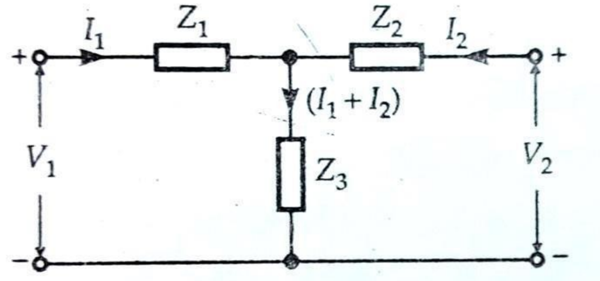
Conditions for Reciprocity and symmetry for Two Port Networks in terms of the various parameters :

Parameter	Condition for reciprocity	Condition for symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{21}$
h	$h_{12} = -h_{21}$	$\Delta h = 1$
$ABCD$	$AD - BC = 1$	$A = D$

- **Relations of Interconnected two port Networks :**
  - *The overall* ***Z parameters*** *for the* ***series*** *connected two port networks are the* ***sum of the Z parameters of the individual two port networks.***
  - *The overall* ***Y parameters*** *for the* ***parallel*** *connected two port networks are the* ***sum of the Y parameters of the individual two port networks****.*
  - *The overall* ***ABCD Parameter matrix*** *of* ***cascaded*** *two Port Networks is the* ***product of the ABCD matrices of the individual networks.***

Illustrative problems :

Example 1: Find the Z Parameters of the following Two Port Network and draw it's equivalent circuit in terms of Z1 Z2 and Z3 .



Solution: Applying KVL to the above circuit in the two loops ,with the current notation as shown, the loop equations for V1 and V2 can be written as :

$$V_1 = I_1 Z_1 + (I_1 + I_2) Z_3$$

or $$V_1 = (Z_1 + Z_3) I_1 + Z_3 I_2 \quad ...(i)$$

and $$V_2 = I_2 Z_2 + (I_2 + I_2) Z_3$$

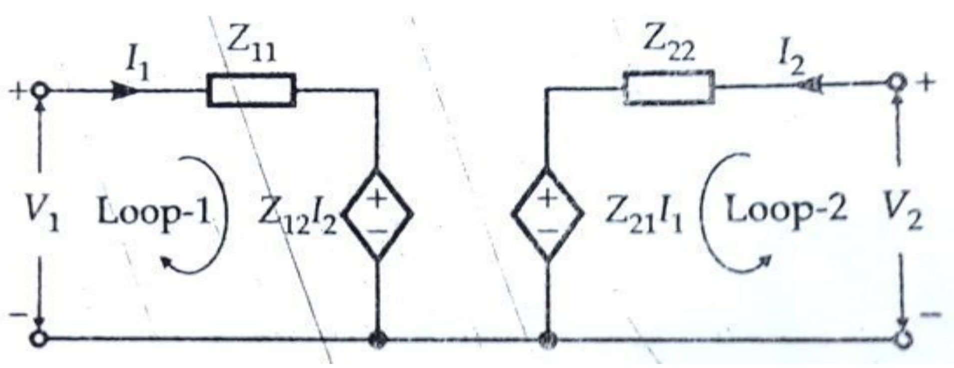
or $$V_2 = Z_3 I_1 + (Z_2 + Z_3) I_2 \quad ...(ii)$$

Comparing the equations (i) and (ii) above with the standard expressions for the Z parameter equations we get :

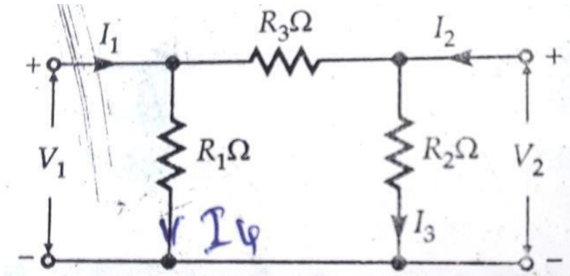
$$Z_{11} = Z_1 + Z_3 ; Z_{12} = Z_3 ;$$

$$Z_{21} = Z_3 ; Z_{22} = Z_2 + Z_3$$

Equivalent circuit in terms of Z1 Z2 and Z3 is shown below.



Example 2: Determine the Z parameters of the **π** type two port network shown in the figure below.



Solution:

From the basic Z parameter equations We know that

$$\mathbf{Z}_{11} = \mathbf{V}_1/\mathbf{I}_1 \Big|\ \mathbf{I}_2=0$$
$$\mathbf{Z}_{12} = \mathbf{V}_1/\mathbf{I}_2 \Big|\ \mathbf{I}_1=0$$
$$\mathbf{Z}_{21} = \mathbf{V}_2/\mathbf{I}_1 \Big|\ \mathbf{I}_2=0$$
$$\mathbf{Z}_{22} = \mathbf{V}_2/\mathbf{I}_2 \Big|\ \mathbf{I}_1=0$$

We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$

1. We can observe that $\mathbf{Z}_{11} = \mathbf{V}_1/\mathbf{I}_1$ with $\mathbf{I}_2=0$ is the parallel combination of R_1 and $(R_2 + R_3)$.

$$\therefore\ \mathbf{Z_{11} = R_1 (R_2 + R_3) / (R_1+R_2 + R_3)}$$

2. $\mathbf{Z_{21} = V_2/I_1 \Big|\ I_2=0}$

By observing the network we find that the current I_1 is dividing into I_3 and I_4 as shown in the figure where I_3 is flowing through R_2(and R_3 also since $I_2=0$)

Hence $V_2 = I_3 \times R_2$

From the principle of current division we find that $I_3 = I_1 . R_1 / (R_1+R_2 + R_3)$

Hence $V_2 = I_3 \times R_2 = [I_1 . R_1 / (R_1+R_2 + R_3)].R_2 = I_1 . R_1 R_2 / (R_1+R_2 + R_3)$

And $V_2/I_1 = R_1 R_2 / (R_1+R_2 + R_3)$

$$\therefore\ \mathbf{Z_{21} = R_1 R_2 / (R_1+R_2 + R_3)}$$

Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$

3. $\mathbf{Z_{12} = V_1/I_2 \Big|\ I_1=0}$

By observing the network we find that the current I_2 is now dividing into I_3 and I_4 as shown in the figure where I_4 is flowing through R_1 (and R_3 also since $I_1 = 0$) Hence $V_1 = I_4 \times R_1$ Again from the principle of current division we find that $I_4 = I_2 . R_2 / (R_1+R_2 + R_3)$

Hence $V_1 = I_4 \times R_1 = [I_2 . R_2 / (R_1+R_2 + R_3)].R_1 = I_2 . R_1 R_2 / (R_1+R_2 + R_3)$

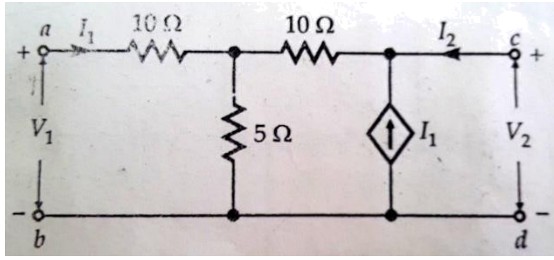
And $V_1/I_2 = R_1 R_2 / (R_1+R_2 + R_3)$

$$\therefore\ \mathbf{Z_{12} = R_1 R_2 / (R_1+R_2 + R_3)}$$

4. We can again observe that $\mathbf{Z}_{22} = \mathbf{V}_2/\mathbf{I}_2$ with $\mathbf{I}_1=0$ is the parallel combination of R_2 and $(R_1 + R_3)$

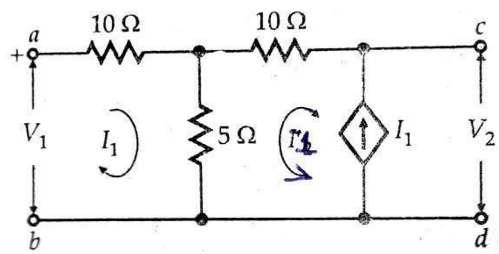
$$\mathbf{Z_{22} = R_2 (R_1 + R_3) / (R_1+R_2 + R_3)}$$

Example 3 : Determine the Z parameters of the network shown in the figure below.



1). We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

With this condition the circuit is redrawn as shown below.



Since the current source is there in the second loop which is equal to I_1 and I_2 is zero, only

current I_1 flows through the right hand side resistance of 10Ω and both currents I_1(both loop currents) pass through the resistance of 5 Ω as shown in the redrawn figure .

Now the equation for loop one is given by :

$V_1 = 10x\, I_1 + 5\,(2\, I_1) = 20\, I_1$ and $V_1/I_1 = 20Ω$

$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 20Ω$

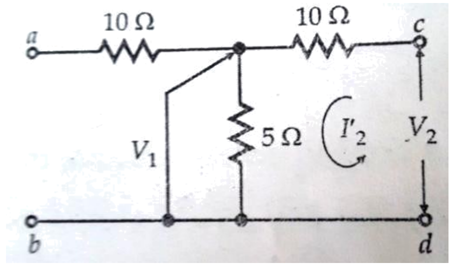
Next the equation for loop two is given by :

$V_2 = 10x\, I_1 + 5\,(2\, I_1) = 20\, I_1$ and $V_2/I_1 = 20Ω$

$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 20Ω$

2). Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$ (input open circuited)

With this condition the circuit is redrawn as shown below.



Now since the current I_1 is zero ,the current source of I_1 would no longer be there in the output loop and it is removed as shown in the redrawn figure. Further since input current $I_1= 0$,there

would be no current in the input side 10Ω and the same current I_2 only flows through common resistance of 5 Ω and output side resistance of 10 Ω .With these conditions incorporated, now

we shall rewrite the two loop equations (for input V_1 and output V_2) to get ***Z_{12}* and *Z_{22}***

Equation for loop one is given by :

$V_1 = 5 I_2$ and $V_1/I_2 = 5\Omega$

$$\therefore \; \mathbf{V}_1/\mathbf{I}_2 \big|_{\mathbf{I}_1=0} = \mathbf{Z}_{12} = 5\Omega$$

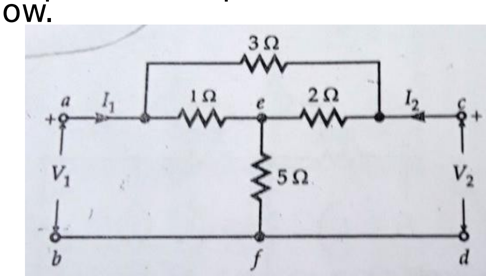
And the equation for loop two is given by:

$V_2 = 10 \times I_2 + 5 \times I_2 = 15 I_2$ and $V_2/I_2 = 15\Omega$

$$\therefore \; \mathbf{V}_2/\mathbf{I}_2 \big|_{\mathbf{I}_1=0} = \mathbf{Z}_{22} = 15\Omega$$

Finally: $\mathbf{Z}_{11} = 20\Omega$; $\mathbf{Z}_{12} = 5\Omega$; $\mathbf{Z}_{21} = 20\Omega$; $\mathbf{Z}_{22} = 15\Omega$

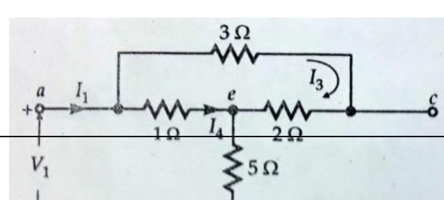
Example 4: Obtain the open circuit parameters of the Bridged T network shown in the figure below.



Open circuit parameters are same as Z parameters.

***1)*. We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)**

With this condition the circuit is redrawn as shown below.



From the inspection of the figure in this condition it can be seen that (since I_2 is zero) the two resistances i.e the bridged arm of 3Ω and output side resistance of 2Ω are in series and

together are in parallel with the input side resistance of 1Ω. Hence the loop equation for V_1 can be written as:

$V_1 = I_1 \times [(3+2) \parallel 1 + 5] = I_1 \times 35/6$ and $V_1/I_1 = 35/6$

$$\left. \frac{V_1}{I_1} \right|_{I_2=0} = Z_{11} = 35/6\Omega$$

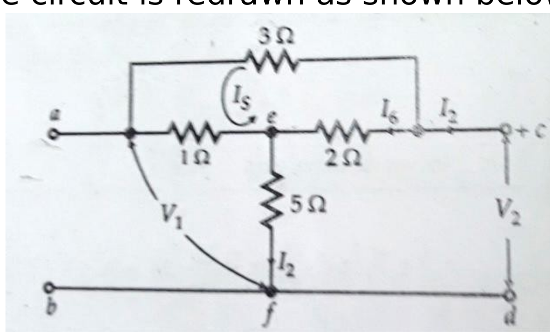
Next the loop equation for V_2 can be written as :

$V_2 = I_3 \times 2 + I_1 \times 5$

But we know from the principle of current division that the current $I_3 = I_1 \times [1/(1+2+3)] = I_1 \times 1/6$ Hence $V_2 = I_1 \times 1/6 \times 2 + I_1 \times 5 = I_1 \times 16/3$ and $V_2 / I_1 = 16/3\ \Omega$

$$\therefore \left. V_2/I_1 \right|_{I_2=0} = Z_{21} = 16/3\ \Omega$$

2). ***Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$ (input open circuited)*** With this condition the circuit is redrawn as shown below.



From the inspection of the figure in this condition it can be seen that (since I_1 is zero) the two resistances i.e the bridged arm of 3Ω and input side resistance of 1Ω are in series and together are in parallel with the output side resistance of 2Ω. Further $I_2 = I_5 + I_6$

Hence the loop equation for V_1 can be written as :

$V_1 = I_5 \times 1 + I_2 \times 5$

But we know from the principle of current division that the current $I_5 = I_2 \times [2/(1+2+3)] = I_2 \times 1/3$ Hence $V_1 = I_2 \times 1/3 \times 1 + I_2 \times 5 = I_2 \times 16/3$ and $V_1 / I_2 = 16/3\ \Omega$

$$\left. V_1/I_2 \right|_{I_1=0} = Z_{12} = 16/3\ \Omega$$

Next the loop equation for V_2 can be written as:

$V_2 = I_6 \times 2 + I_2 \times 5$

But we know from the principle of current division that the current $I_6 = I_2 \times [1/(1+2+3)] = I_2 \times (3+1)/6 = (I_2 \times 2/3)$

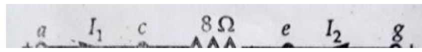
Hence $V_2 = I_2 \times (2/3) \times 2 + I_2 \times 5 = I_2 \times 19/3$

$= 19/3$

and V_2/I_2

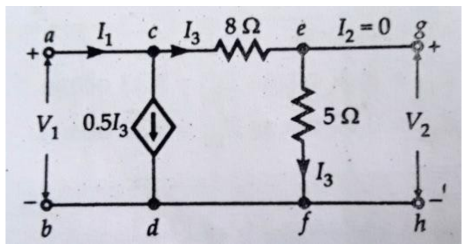
$$\left. \mathbf{V/I_2} \right|_{\mathbf{I=0}} = \mathbf{Z_{22} = 19/3\ \Omega}$$

Example 5 : Obtain Z parameters of the following **π** network with a controlled current source of 0.5 I_3 in the input port.



1). We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

With this condition the circuit is redrawn as shown below.



In this condition we shall first apply Kirchhoff's current law to the node **'c':**

Then $I_1 = 0.5I_3 + I_3$ (I_3 being the current through the resistances of 8 Ω and 5 Ω) i.e $I_1 = 0.5I_3 + I_3$ or $I_1 = 1.5I_3$ or $I_3 = I_1/1.5$ i.e $I_3 = (2/3)I_1$ Now we also observe that $V_1 = I_3(8+5) = 13.\ I_3$

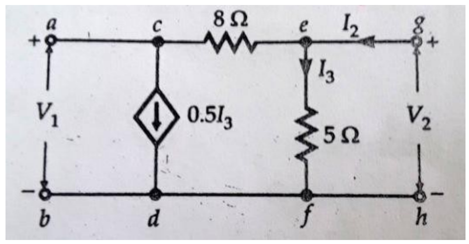
Using the value of $I_3 = (2/3)I_1$ into the above expression we get $V_1 = 13(2/3)I_1$ and $V_1/\ I_1 = 26/3 = 8.67$

$$\mathbf{V_1/I_1 \Big|_{I_2=0} = \quad Z_{11} = 8.67\Omega}$$

Next we also observe that $V_2 = 5 \cdot I_3$ and substituting the above value of $I_3 = (2/3)I_1$ into this expression for V_2 we get :
$V_2 = 5 \cdot I_3$ i.e $V_2 = 5 \cdot (2/3)I_1$ i.e $V_2 / I_1 = 10/3 = 3.33\Omega$

$$\mathbf{V_2/I_1 \big|_{I_2=0} = \; Z_{21} = 3.33\ \Omega}$$

2). Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$ (input open circuited)
With this condition the circuit is redrawn as shown below.



In this condition now we shall first apply Kirchhoff's current law to the node **'e':**

Then $I_2 = 0.5I_3 + I_3$ ($0.5.I_3$ being the current through the resistance of 8 Ω and I_3 being the current through the resistances of 5 Ω)
i.e $I_2 = 0.5I_3 + I_3$ or $I_2 = 1.5I_3$ or $I_3 = I_2/1.5$ i.e $I_3 = (2/3)I_2$
Now we also observe that $V_1 = (-0.5I_3 \times 8 + I_3 x5) = I_3$ (it is to be noted here carefully that – sign

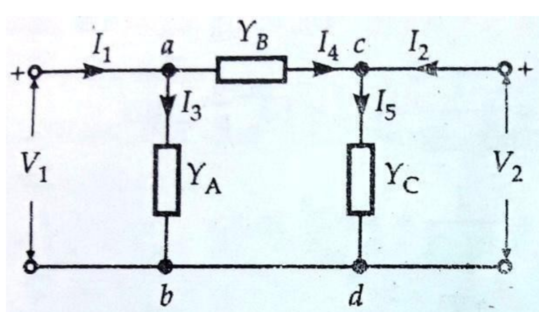
is to be taken before $0.5I_3$ x8 since the current flows through the resistance of 8 Ω now in the reverse direction.
Using the value of $I_3 = (2/3)I_2$ into the above expression for V_1 we get $V_1 = (2/3)I_2$ and $V_1/ I_2 = 0.67$

$$\therefore \mathbf{V_1/I_2 \big|_{I_1=0} = \quad Z_{12} = 0.67\Omega}$$

Next we also observe that $V_2 = 5 \cdot I_3$ and substituting the above value of $I_3 = (2/3)I_2$ into this expression for V_2 we get :
$V_2 = 5 \cdot I_3$ i.e $V_2 = 5 \cdot (2/3)I_2$ i.e $V_2 / I_2 = 10/3 = 3.33\Omega$

$$\mathbf{V/I \big|_{I_1=0} = \quad Z_{21} = 3.33\ \Omega}$$

Example 6 : Find the Y parameters of the following **π** type two port network and draw it's Y parameter equivalent circuit in terms of the given circuit parameters.



Applying KCL at node (a) we get

$$I_1 = I_3 + I_4$$
$$I_1 = V_1 Y_A + (V_1 - V_2) Y_B$$
$$I_1 = V_1 (Y_A + Y_B) + (-Y_B) V_2 \qquad \text{--(i)}$$

Similarly applying KCL to node (c) we get

$$I_2 = I_5 - I_4$$
$$I_2 = V_2 Y_C - (V_1 - V_2) Y_B$$
$$I_2 = (-Y_B) V_1 + (Y_C + Y_B) V_2 \qquad \text{...(ii)}$$

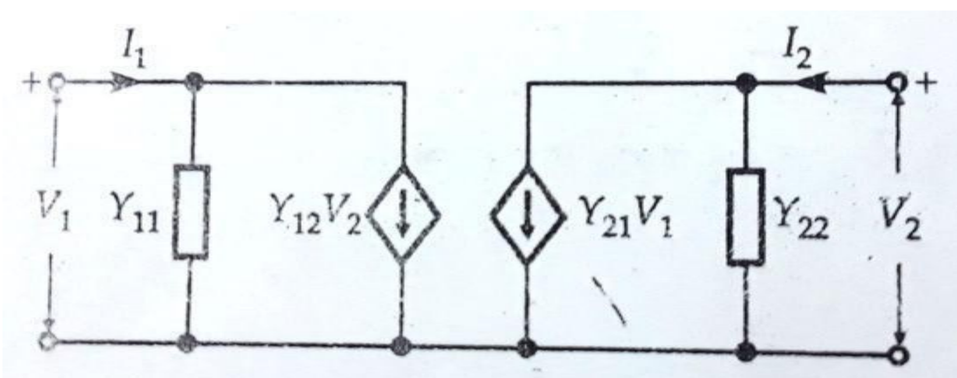
Comparing the equations (i) and (ii) above with the standard expressions for the Y parameter equations we get :

$$Y_{11} = (Y_A + Y_B); \; Y_{12} = -Y_B$$
$$Y_{21} = -Y_B; \; Y_{22} = Y_C + Y_B.$$

Observing the equations (i) and (ii) above we find that :

- The terms $V_1 (Y_A + Y_B)$ and $V_2(Y_C+Y_B)$ are the currents through the admittances Y_{11} and Y_{22} and
- The terms $-Y_B .V_2$ and $-Y_B .V_1$ are the dependent current sources in the input and the output ports respectively.

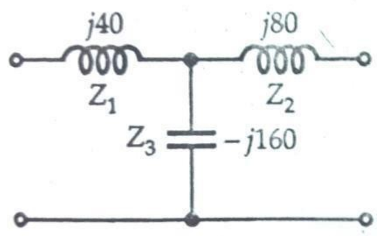
These observations are reflected in the equivalent circuit shown below.



In the above figure $Y_{11} = (Y_A+ Y_B)$ & $Y_{22} = (Y_C+Y_B)$ are the admittances and

$Y_{12} .V_2 = -Y_B .V_2$ & $Y_{21} .V_1 = -Y_B .V_1$ are the dependent current sources

Example 7: Find the Y parameters of the following network



Solution: We will solve this problem in two steps.

1. We shall first express the Z parameters of the given T network in terms of the impedances Z1, Z2 and Z3 using the standard formulas we already know and substitute the given values of Z1, Z2 and Z3 .

$$Z_{11} = Z_1 + Z_3 = -j\,120\,;$$
$$Z_{12} = Z_3 = -j\,160$$
$$Z_{21} = Z_3 = -j\,160\,;$$
$$Z_{22} = Z_2 + Z_3 = -j\,80$$

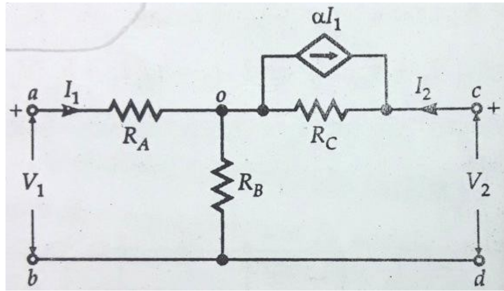
2. Then convert the values of the Z parameters into Y parameters i.e express the Y parameters in terms of Z parameters using again the standard relationships.

$$Y_{11} = \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}}$$
$$= \frac{-j\,80}{(-j\,120)(-j\,80) - (-j\,160)^2}$$
$$= \frac{-j\,80}{16{,}000} = \frac{\mathbf{-j}}{\mathbf{200}}\ \textbf{mho.}$$

$$Y_{12} = Y_{21} = \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}Z_{21}}$$
$$= \frac{j\,160}{16{,}000} = \frac{\mathbf{j}}{\mathbf{100}}\ \textbf{mho.}$$

$$Y_{22} = \frac{Z_{11}}{Z_{11}Z_{22} - Z_{12}Z_{21}}$$
$$= \frac{-j\,120}{16{,}000} = \frac{\mathbf{-j}}{\mathbf{133.33}}\ \textbf{mho.}$$

Example 8: Find the **‘ h’** parameters of the network shown below. (fig12.34)



First let us write down the basic **‘ h’** parameter equations and give the definitions of the **‘ h’** parameters.

$$V_1 = h_{11}.\ I_1 + h_{12}.V_2$$

$$I_2 = h_{21}.I_1 + h_{22}.V_2$$

h_{11} = V_1/I_1 with V_2 =0 **h_{21} = I_2/I_1 with V_2 =0**

h_{12} = V_1/V_2 with I_1=0 **h_{22} = I_2/V_2 with I_1=0**

Now

1). ***We will first find out h_{11} and h_{21} which are given by the common condition V_2 = 0 (Output short circuited)***

In this condition it can be observed that the resistance R_C and the current source αI_1 become parallel with resistance R_B.

For convenience let us introduce a temporary variable **V** as the voltage at the node **‘o’.** Then the current through the parallel combination of R_B and R_C would be equal to

$$\frac{V}{\dfrac{R_B\ R_C}{R_B + R_C}} = \frac{V(R_B + R_C)}{R_B\ R_C}$$

Then applying KCL at the node **‘o’** we get

$$I_1 = \frac{V(R_B + R_C)}{R_B\ R_C} + \alpha\ I_1$$

$$I_1(1-\alpha) = \frac{V(R_B + R_C)}{R_B\ R_C}$$

$$\therefore \quad V = \frac{(1-\alpha)\ I_1\ R_B\ R_C}{(R_B + R_C)}$$

Next applying KVL at input port we get $V_1 = I_1.R_A + V$ and $V_1/I_1 = R_A + V / I_1$ Now using the value of V we obtained above in this expression for V_1/I_1 we get

$$h_{11} = \frac{V_1}{I_1} = R_A + \frac{(1-\alpha)\, R_B\, R_C}{R_B + R_C}$$
$$= \frac{R_A\,(R_B + R_C) + (1-\alpha) R_B\, R_C}{R_B + R_C} \text{ ohm.}$$

Again from inspection of the figure above it is evident that

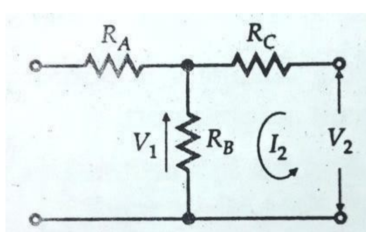
$$I_2 = -\left(\alpha\, I_1 + \frac{V}{R_C}\right)$$
$$I_2 = -\alpha I_1 - \frac{(1-\alpha)\, I_1\, R_B}{(R_B + R_C)}$$

Therefore

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0} = -\alpha - \frac{(1-\alpha)\, R_B}{(R_B + R_C)}$$
$$= -\frac{(\alpha\, R_C + R_B)}{(R_B + R_C)}.$$

2). *Next we will find out h_{12} and h_{22} which are given by the common condition $I_1 = 0$ (Input open circuited)*

Now since I_1 is zero , the current source disappears and the circuit becomes simpler as shown in the figure below.

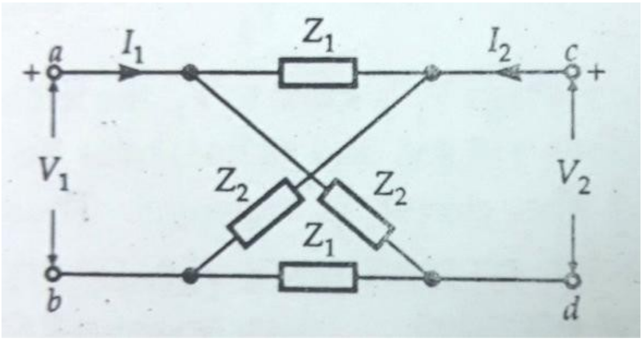


Now applying KVL at the output port we get:

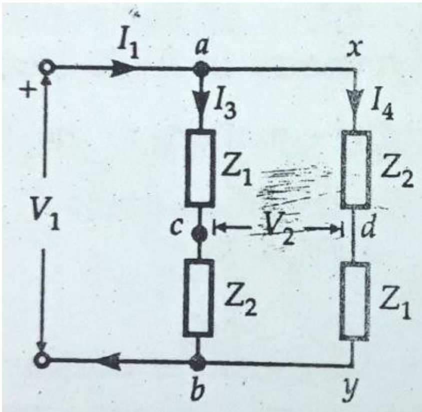
$$V_2 = I_2\,(R_B + R_C)$$
$$\left.\frac{I_2}{V_2}\right|_{I_1 = 0} = h_{22} = \left(\frac{1}{R_B + R_C}\right) \text{mho.}$$

Again under this condition:

Example 9 : Z parameters of the lattice network shown in the figure below.



First we shall redraw the given lattice network in a simpler form for easy analysis as shown below.



We will then find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

It can be observed that the impedances in the two arms **'ab'** and **'xy'** are same i.e $Z_1 + Z_2$ and
their parallel combination is $(Z_1 + Z_2)/2$
Hence applying KVL at the input port we get

$$V_1 = I_1\left(\frac{Z_1 + Z_2}{2}\right)$$

$$\frac{V_1}{I_1} = Z_{11}\Big|_{I_2=0} = \frac{Z_1 + Z_2}{2}$$

Next we find that

$$V_2 = V_c - V_d = (V_1 - I_3 Z_1) - (V_1 - I_4 Z_2)$$
$$= I_4 Z_2 - I_3 Z_1$$

($\mathbf{V_C}$ and $\mathbf{V_D}$ being the potentials at points '**c'** and '**d')**

It can also be observed from the simplified circuit that the currents I_3 and I_4 through the branches **'ab'** and **'xy'** are equal since the branch impedances are same and same voltage V_1 is

applied across both the branches. Hencethe current I divides equally as I_3 and I_4 i.e $I_3 = I_4 = I/2$ Now substituting these values of I_3 and I_4 in the expression for V_2 above:

$$V_2 = \frac{I_1}{2} \times Z_2 - \frac{I_1}{2} \times Z_1 = \frac{Z_2 - Z_1}{2} . I_1$$

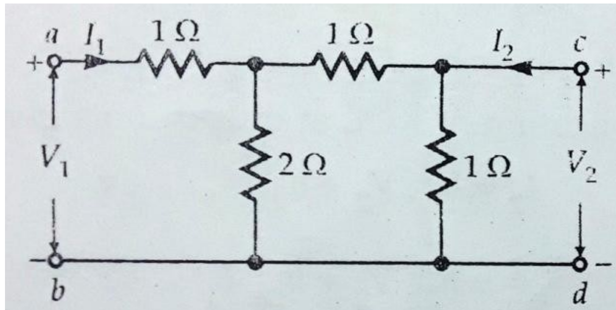
$$\frac{V_2}{I_1} = Z_{21}\Big|_{I_2=0} = \frac{Z_2 - Z_1}{2}$$

As can be seen the circuit is both symmetrical and Reciprocal and hence :

$$Z_{11} = Z_{22} = \frac{Z_1 + Z_2}{2}$$

$$Z_{12} = Z_{21} = \frac{Z_2 - Z_1}{2}.$$

Example 10: Find the transmission parameters of the following network (fig 12.51)



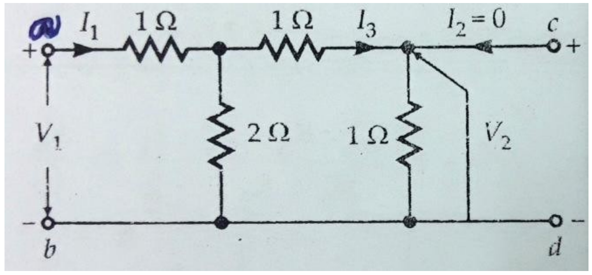
First let us write down the basic ABCD parameter equations and give their definitions.

$$\mathbf{V_1 = A.V_2 - B.I_2}$$
$$\mathbf{I_1 = C.V_2 - D.I_2}$$

A = V_1/V_2	**with I_2**	**= 0**
C = I_1/V_2	**with I_2**	**= 0**
B = $V_1/-I_2$	**with V_2 = 0**	
D = $I_1/-I_2$	**with V_2 = 0**	

1).We will then find out A and C which are given by the common condition I2 = 0 (Output open circuited)

The resulting circuit in this condition is redrawn below.



Applying KVL we can write down the two mesh equations and get the values of A and C :

$$V_1 = I_1 \times 1 + (I_1 - I_3)2$$

or $$V_1 = 3I_1 - 2I_3 \qquad ...(i)$$

and $$0 = (I_3 - I_1)2 + I_3(1+1) = 4I_3 - 2I_1$$

$$\therefore \quad I_3 = \frac{1}{2} I_1 \qquad ...(ii)$$

Utilising (*ii*) in (*i*),

$$V_1 = 3I_1 - 2 \times \frac{1}{2} I_1 = 2I_1 \qquad ...(iii)$$

$\therefore$ Again,

$$V_2 = I_3 \times 1 = \frac{1}{2} I_1 \qquad ...(iv)$$

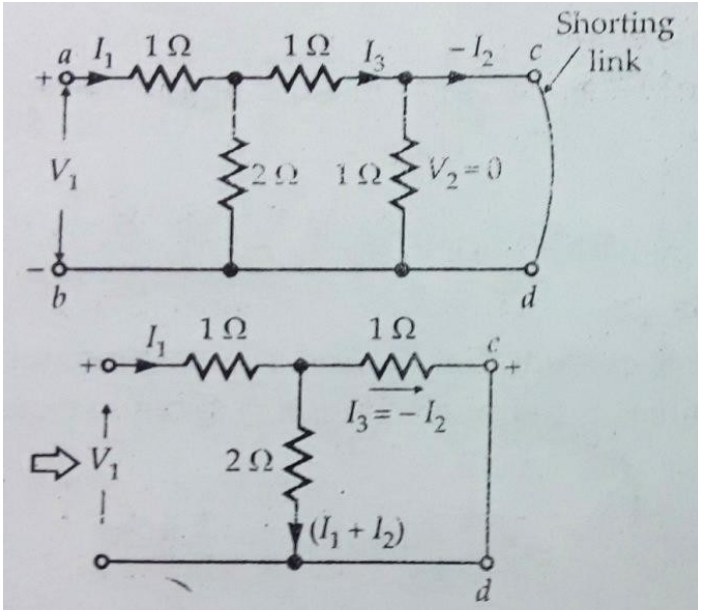
$$\therefore \quad \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = 2 \text{ mho} = C.$$

Dividing equation (*iii*) by (*iv*),

$$\left.\frac{V_1}{V_2}\right|_{I_2 = 0} = 4 = A.$$

2.) ***Next we will find out B and D which are given by the common condition V2 = 0 (Output short circuited*)**

The resulting simplified network in this condition is redrawn below.



The voltage at the input port is given by : $V_1 = I_1 \times 1 + (I_1 + I_2) \times 2$ i.e. $V_1 = 3I_1 + 2I_2$ .... (i)

And the mesh equation for the closed mesh through ‘cd’ is given by :

$0 = I_2 \times 1 + (I_1 + I_2) \times 2$ or $3 I_2 + 2 I_1 = 0$ or

$I_1 = -(3/2).\ I_2$ .... (ii)

Using equation (ii) in the equation (i) above we get :

$V_1 = -(9/2)\ I_2 + 2I_2 = -(5/2)I_2$

Or $V_1 / -I_2 = B = (5/2)$

And from equation (ii) above we can directly get

$I_1 / -I_2 = D = 3/2$

Hence the transmission parameters can be written in matrix notation as :

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4 & \frac{5}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$$

Here we can see that $AD - BC = 1$ and $A \neq D$

Hence the network is Symmetrical but not Reciprocal.

UNIT-3

FILTERS AND ATTENUATORS

Classification of
filters Filter networks

Classification of pass band and stop
band Characteristic Impedance in the
pass and stop bands

Constant ‘k’ low pass filter
Constant ‘k’ high pass filter
‘m’ derived ‘T’ section

‘m’ derived low pass & high pass filters

Band pass filter
Band Stop filter

Symmetrical Attenuators

‘T’ type attenuators
‘π’ type attenuators

Lattice attenuator

Bridged ‘T’ Attenuator

Important concepts and formulae
Illustrative problems

Previous year examination questions

INTRODUCTION:

A filter is a reactive Network which passes a desired band of frequencies called Pass band without any attenuation and suppresses other band of frequencies called stop band practically with full attenuation. The frequency which separates the stop and passes bands is called the Cutoff frequency f_c

An Ideal filter would pass the desired band of frequencies without any attenuation and suppresses the stop band of frequencies with full attenuation. Further it would have a sharp cutoff frequency i.e. the rise and fall of the attenuation characteristic would be very steep and almost vertical.

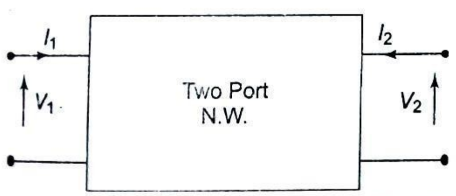
But practical filters would have a finite attenuation during the pass band and would not have full attenuation in the stop band due to losses in the circuit elements. Further the rise and fall of the attenuation characteristics would also have finite rise and fall times.

The important properties of the filters are given below and all these properties will be discussed in detail.

- Characteristic Impedance
- Frequency response or attenuation characteristic (The attenuation in Pass and stop bands)
  - Cutoff frequency (the frequency of transition from Stop to Pass band and vice versa) characteristic

Decibel and Neper:

The attenuation of a filter can be expressed in Decibels or Nepers. A Neper is defined as the Natural logarithm of the ratio of input voltage (or current) to output voltage (or current) when the network is properly terminated in its Characteristic Impedance. For the two port network shown in the figure below.



The number of Nepers N will be :

$$N = \log_e \left|\frac{V_1}{V_2}\right| \text{ or } \log_e \left|\frac{I_1}{I_2}\right|$$

When ratio of input Power to Out power is expressed in Nepers the definition becomes

A decibel is defined as ten times the common logarithm of the ratio of the Input Power to the Output Power.

Therefore Decibel

$$D = 10 \log_{10} \frac{P_1}{P_2}$$

When the ratios of Voltages or Currents are expressed in decibels the definition of decibel becomes:

$$D = 20 \log_{10} \left|\frac{V_1}{V_2}\right| = 20 \log_{10} \left|\frac{I_1}{I_2}\right|$$

One Decibel = 0.115 Nepers

CLASSIFICATION OF FILTERS:

Filters are classified depending on the Frequency Characteristic and also depending upon the impedances in the series and parallel arms (Series arm Impedance Z_1 and Parallel arm impedance Z_2).

Classification based on the frequency characteristics:

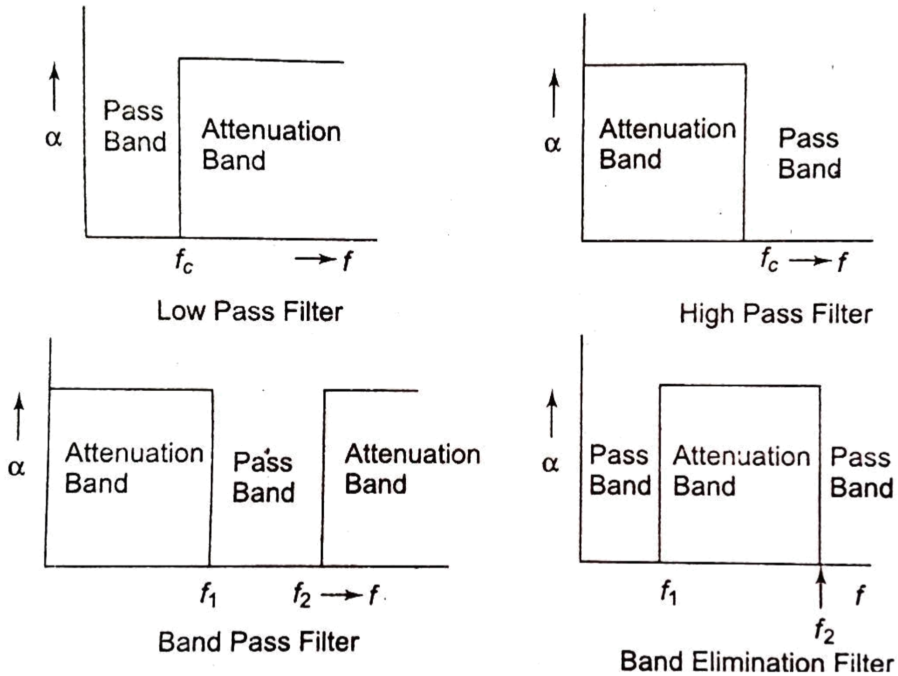
Low pass filter: Is one which passes all frequencies without any attenuation up to a cutoff frequency f_c and attenuates frequencies higher than f_c . The frequency band from 0 to f_c is called Pass band or transmission band. The frequency band beyond f_c is called attenuation band.

High pass filter : Is one which passes all frequencies without any attenuation beyond a cutoff frequency f_c and attenuates frequencies lower than f_c . The frequency band from 0 to f_c is called attenuation band. The frequency band beyond f_c is called Pass band or transmission band.

Band Pass filter: Is one which passes all frequencies without any attenuation between two designated frequencies f_1 & f_2 and attenuates frequencies lower than f_1 & higher than f_2 . The frequency f_1 is called the lower cut-off frequency and the frequency f_2 is called the upper cut-off frequency.

Band stop or elimination filter: Is one which attenuates all frequencies between two designated frequencies f_1 & f_2 and passés without any attenuation frequencies lower than f_1 & higher than f_2 .

The frequency characteristics of all these four types of filters are shown below.



Classification based on the relationship between impedances in the series and parallel arms (Series arm
Impedance Z_1 and Parallel arm impedance Z_2):

Constant 'k' filter (Prototype filter): In this filter the series arm and shunt arm impedances (Z_1 and Z_2)

are such that : $Z_1 . Z_2 = R_0^2 = K$ (a constant) where R_0 is a real number and is independent of frequency. It is also known the design Impedance. A filter which is designed with this relation is called a constant 'k' filter or a prototype filter. They can be further classified based on the frequency characteristics as constant 'k ' type low pass ,high pass etc.

FILTER NETWORKS:

Filters should have ideally zero attenuation in the pass band. This condition can be achieved if the filter elements are dissipation less. To achieve this filters are designed with reactive elements like Inductors and capacitors which are ideally dissipation less. Filters are made of symmetrical **T** or **π** sections which can be considered as combinations of L sections as shown in the figure below.

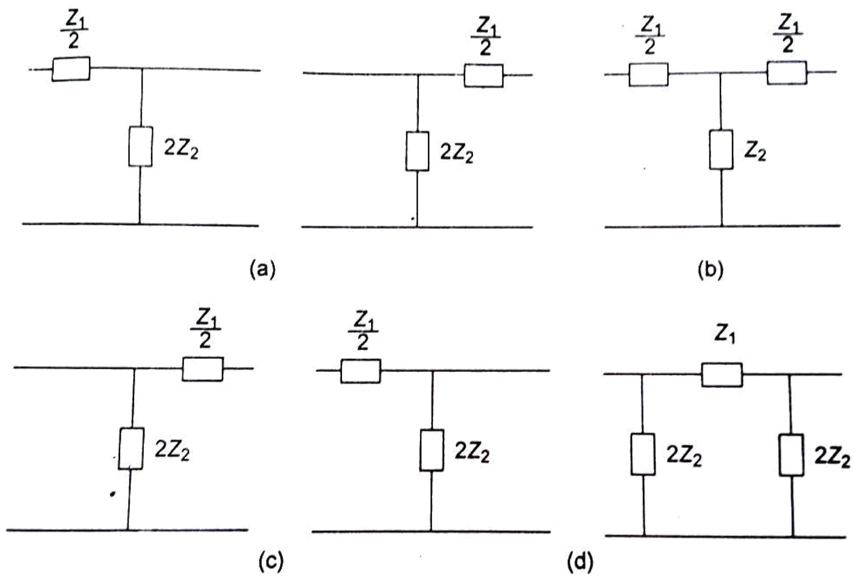


Fig: Combination of L sections. (a) and (b) To form a T section. (c) and (d) to form a π section

CLASSIFICATION OF PASS BAND AND STOP BAND:

So far we have studied briefly about the T and π types of filters and different characteristics of filters. Before we take up study ,analysis and design of these filters with **L** and **C** components we have to study about *Characteristic Impedance* of **T** and **π** types of filters.

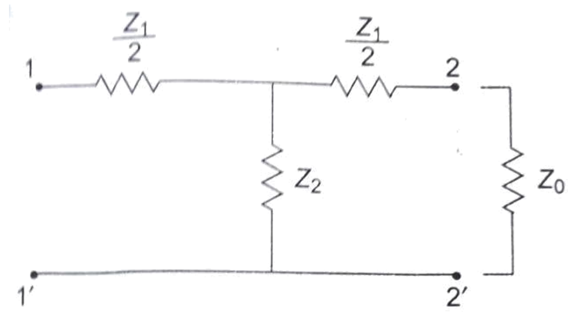
CHARACTERISTIC IMPEDANCE OF T AND Π NETWORKS:

Characteristic impedance of a two port Network Z_0 is defined as that Impedance which when terminated across the output terminals of a TPNW will result in the same Z_0 as the input impedance.

Defined in another way, if the image impedances at the Input and output ports of a TPNW are identical then the Image impedance is also called the Characteristic or Iterative Impedance.

Let us now derive the Characteristic Impedances for T and π networks from the above definitions:

T Network: Let us consider a symmetrical T network as shown in the figure below.



The value of the Input impedance for this network when it is terminated by Z_0 is given by :

$$Z_{in} = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_0\right)}{\frac{Z_1}{2} + Z_2 + Z_0}$$

But by definition of Characteristic impedance

$$Z_{in} = Z_0$$

Therefore

$$Z_0 = \frac{Z_1}{2} + \frac{2Z_2\left(\frac{Z_1}{2} + Z_0\right)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1}{2} + \frac{(Z_1Z_2 + 2Z_2Z_0)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1^2 + 2Z_1Z_2 + 2Z_1Z_0 + 2Z_1Z_2 + 4Z_0Z_2}{2(Z_1 + 2Z_2 + 2Z_0)}$$

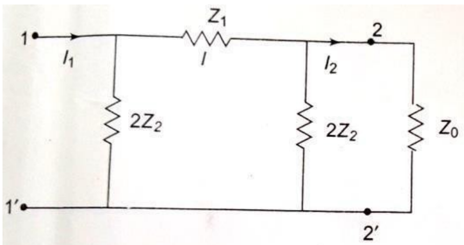
$$4Z_0^2 = Z_1^2 + 4Z_1Z_2$$

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2$$

Thus the characteristic impedance of a symmetrical T NW is given by:

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

π Network : : Let us consider a symmetrical π network as shown in the figure below:



The value of the Input impedance for this network when it is terminated by Z_0 is given by :

$$Z_{in} = \frac{2Z_2\left[Z_1 + \frac{2Z_2\,Z_0}{2Z_2 + Z_0}\right]}{Z_1 + \frac{2Z_2\,Z_0}{2Z_2 + Z_0} + 2Z_2}$$

But by definition of Characteristic impedance

$$Z_{in} = Z_0$$

Therefore

$$Z_0 = \frac{2Z_2\left[Z_1 + \frac{2Z_2\,Z_0}{2Z_2 + Z_0}\right]}{Z_1 + \frac{2Z_2\,Z_0}{2Z_2 + Z_0} + 2Z_2}$$

$$Z_0Z_1 + \frac{2Z_2Z_0^2}{2Z_2 + Z_0} + 2Z_0Z_2 = \frac{2Z_2(2Z_1Z_2 + Z_0Z_1 + 2Z_0Z_2)}{(2Z_2 + Z_0)}$$

$$2Z_0Z_1Z_2 + Z_1Z_0^2 + 2Z_0^2Z_2 + 4Z_2^2Z_0 + 2Z_2Z_0^2$$
$$= 4Z_1Z_2^2 + 2Z_0Z_1Z_2 + 4Z_0Z_2^2$$

$$Z_1Z_0^2 + 4Z_2Z_0^2 = 4Z_1Z_2^2$$

$$Z_0^2(Z_1 + 4Z_2) = 4Z_1Z_2^2$$

$$Z_0^2 = \frac{4Z_1Z_2^2}{Z_1 + 4Z_2}$$

Rearranging the above equation leads to

$$Z_0 = \sqrt{\frac{Z_1Z_2}{1 + Z_1 / 4Z_2}}$$

Further simplification gives the characteristic Impedance of a π Network as

$$Z_{0\pi} = \frac{Z_1Z_2}{\sqrt{Z_1Z_2 + Z_1^2 / 4}}$$

But we know that

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2}$$

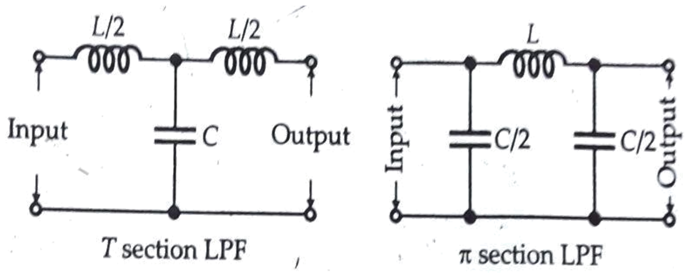
From this we get a relationship between the characteristic Impedances of T and π networks as :

$$Z_{0\pi} = \frac{Z_1Z_2}{Z_{0T}}$$

CONSTANT 'K' LOW PASS FILTER:

Analysis:

The figure below represents T and π networks of a low pass filter.



This filter allows lower order frequencies to pass through and stops higher order frequencies. In both the T and π networks the total series Impedance is given by:

$$Z_1 = j\omega L \qquad [\because X_1 = \omega L]$$

And the shunt impedance is given by

$$Z_2 = \frac{1}{j\omega C} = -\frac{j}{\omega C} \qquad \left[\because X_2 = -\frac{1}{\omega C}\right]$$

Multiplying the above two equations we get

$$Z_1 Z_2 = j\omega L\left(-\frac{j}{\omega C}\right) = \frac{L}{C} = R_0^2 \text{ (say)}$$

(L/C being a real quantity)

And again by using the same two equations we get

$$\frac{Z_1}{4Z_2} = -\frac{\omega^2 LC}{4}$$

But we know that for a T type two port network the characteristic Impedance Z_{0T} is given by

$$Z_{0T} = \sqrt{Z_1 Z_2\left(1+\frac{Z_1}{4Z_2}\right)}$$

Now substituting the above two values of $Z_1.Z_2$ and $Z_1/4Z_2$ in the above equation for Z_{0T} we get

$$Z_{0T} = \sqrt{\frac{L}{C}}\sqrt{1-\frac{\omega^2 LC}{4}}$$

$$= R_0\sqrt{1-\frac{\omega^2 LC}{4}} = R_0\sqrt{1-\frac{\omega^2}{4/LC}}$$

Or

$$Z_{0T} = R_0\sqrt{1-\frac{\omega^2}{\omega_C^2}} \qquad \left[\text{where } \omega_C^2 = \frac{4}{LC}\right]$$

$$= R_0\sqrt{1-\left(\frac{f}{f_C}\right)^2}$$

Thus it can be seen from the above that

$$Z_{0T} = \left(= R_0\sqrt{1-\frac{\omega^2 LC}{4}}\right)$$

is real and the Characteristic Impedance represents the Pass band if $\boldsymbol{\omega^2 LC/4 < 1}$ Z_{OT} is imaginary and represents the stop band if $\boldsymbol{\omega^2 LC/4 > 1}$

It can also be seen that:

When Z_{OT} is real f is less than f_c and
When Z_{OT} is imaginary f is greater than f_c

And the cut-off frequency will be at a condition when $\frac{\omega^2 LC}{4} = 1 \left[i.e., \omega_C = \frac{2}{\sqrt{LC}}\right]$. Hence the LPF (Low Pass Filter)cut-off frequency ω_c is given by :

$$\omega_C = \frac{2}{\sqrt{LC}} \quad \text{or} \quad f_C = \frac{1}{\pi\sqrt{LC}}$$

From the above equation for Z_{OT} it is clear that in the Pass band (when Z_{OT} is real) $f < f_c$ and in the stop band (when Z_{OT} is imaginary) $f > f_c$.

For a π type of Filter the characteristic impedance is given by:

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{R_0^2}{R_0\sqrt{1-\left(\frac{f}{f_C}\right)^2}} = \frac{R_0}{\sqrt{1-\left(\frac{f}{f_C}\right)^2}}$$

Here also in the Pass band $f < f_c$ (so that Z_{OT} is real) and in the stop band $f > f_c$ (so that Z_{OT} is imaginary)

For the π network also

When Z_{OT} is real f is less than f_c and
When Z_{OT} is imaginary f is greater than f_c

LPF cutoff frequency is given by

$$\omega_C = \frac{2}{\sqrt{LC}} \quad \text{or} \quad f_C = \frac{1}{\pi\sqrt{LC}}$$

Figure below shows the Z_{OT} and $Z_{O\pi}$ characteristics of the LPF in the Pass band. It can be seen that $Z_{O\pi}$ increases with frequency and Z_{OT} decreases with frequency in the pass band but both are real.

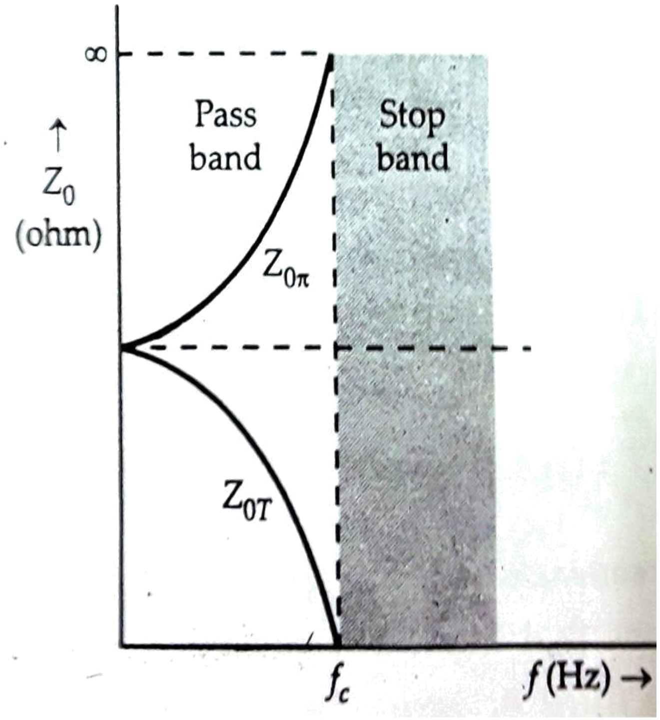


Fig: Characteristic Impedance profile of T and π sections of a LPF

Design:

One of the most important aspects of filter design is selection of the values of components of L and C given the values of design Impedance R_0 and cutoff frequency f_c. In the previous analysis we have obtained the expressions for these two terms as :

$$R_0 = \sqrt{\frac{L}{C}}$$

$$f_C = \frac{1}{\pi\sqrt{LC}}$$

From these two expressions we can get the values of L and C in terms of design Impedance R_0 and cutoff frequency f_c as

$$L = \frac{R_0}{\pi f_c}$$

$$C = \frac{1}{\pi R_0 \ f_C}$$

CONSTANT ‘K’ HIGH PASS FILTER:

Analysis:

The figure below represents T and π sections of a high pass filter.

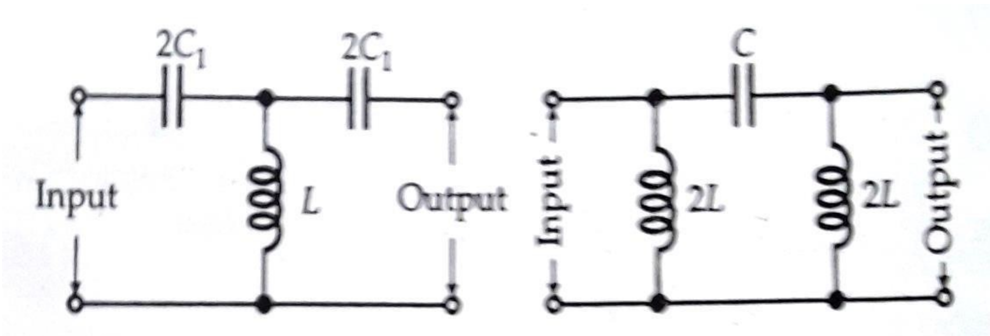


Fig: T and π sections of a constant ‘K’ high pass filter

This filter allows higher order frequencies to pass through and stops lower order frequencies. In both the T and π networks the total series Impedance is given by:

$$Z_1 = \frac{1}{j\omega C}$$

And the shunt impedance is given by : $Z_2 = j\omega L$

Hence

$$Z_1 Z_2 = \frac{1}{j\omega C} \times j\omega L = \frac{L}{C} = R_0^2$$

The Characteristic Impedance of a T section is given by: $Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$

Substituting the above values of Z_1 and $Z_1 Z_2$ into this we get $Z_{0T} = \sqrt{-\frac{1}{4\omega^2 C^2} + \frac{L}{C}}$

$$= \sqrt{\frac{L}{C}}\sqrt{1-\frac{1}{4\omega^2 CL}} \qquad\qquad = R_0\sqrt{1-\frac{1}{4\omega^2 LC}}$$

From the above equations we can see that

If $4\omega^2 LC > 1$ then Z_{0T} is real and the filter works in the pass band

If $4\omega^2 LC < 1$ then Z_{0T} is imaginary and the filter works in the stop band The cutoff frequency is given by $4\omega^2 LC = 1$

$$i.e., \quad \omega_C = \frac{1}{2\sqrt{LC}} \qquad \text{or} \qquad f_c = \frac{1}{4\pi\sqrt{LC}}$$

Using this value of ω_c in the expression for Z_{0T} we get

$$Z_{0T} = R_0\sqrt{1-\frac{1}{4\omega^2 LC}} \qquad = R_0\sqrt{1-\frac{\omega_C^2}{\omega^2}} = R_0\sqrt{1-\frac{f_C^2}{f^2}}$$

Where $\omega_C^2 = 1/4LC$.

Now using the known relation $Z_{0T} \cdot Z_{0\pi} = R_0^2$ we can get the expression for $Z_{0\pi}$ as below:

$$Z_{0\pi} = \frac{R_0^2}{Z_{0T}} = \frac{R_0^2}{R_0\sqrt{1-\frac{\omega_C^2}{\omega^2}}} = \frac{R_0}{\sqrt{1-\frac{\omega_C^2}{\omega^2}}} \qquad = \frac{R_0}{\sqrt{1-\frac{f_C^2}{f^2}}}$$

These two characteristic Impedances of T and π sections are shown in the figure below.

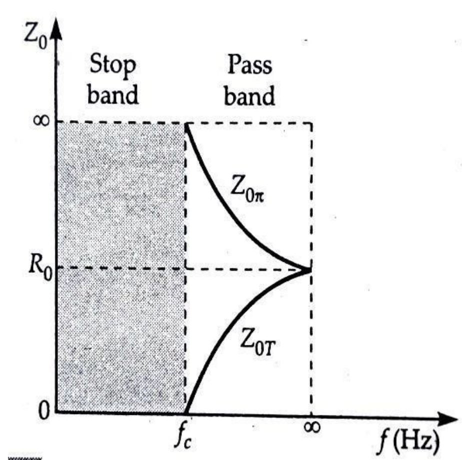


Fig: Characteristic Impedance Profiles of a HPF with T and π sections

Design:

Given the design Impedance R_0 and the cutoff frequency f_c the values of L and C can be obtained from the following two equations derived in the analysis.

$$R_0 = \sqrt{\frac{L}{C}}$$ and $$f_C = \frac{1}{4\pi\sqrt{LC}}$$

Solving these two equations we get the values of L and C as:

$$L = \frac{R_0}{4\pi f_C}$$ and $$C = \frac{1}{4\pi R_0 f_C}$$

‘m’ derived filter :

Introduction:

There are two disadvantages of constant ‘k’ filter:

- The attenuation does not change rapidly beyond the cutoff frequency.
- Characteristic Impedance varies widely in the pass band from the desired value.

In order to overcome these two limitations a new type filter called ‘m’ derived filter has been developed. In this configuration the filter will have a faster rate of change of attenuation outside the pass band but it will have the same type of varying characteristic Impedance throughout the pass band as that of the constant ‘k’ filter. i.e. in this also it is not possible to have constant Impedance throughout the pass band.

A ‘m’ derived filter is identical to the constant ‘k’ type filter except that

In a T section the series arm impedance is	***multiplied*** by the constant **‘m’** and
In a π section the shunt arm impedance is	***divided*** by the constant **‘m’**

Where the value of ‘m’ lies between 0 to 1. Inclusion of ‘m’ in the constant ‘k’ type filter modifies it in such a way that it improves the rate of change of attenuation but the characteristic Impedance continues to vary the same way in the pass band.

‘m’derived ‘ T’ section:

A constant ‘k’ type ‘T’ section and a ‘m’ derived filter with a suitably modified ‘T’ section configuration are shown in the figure below.

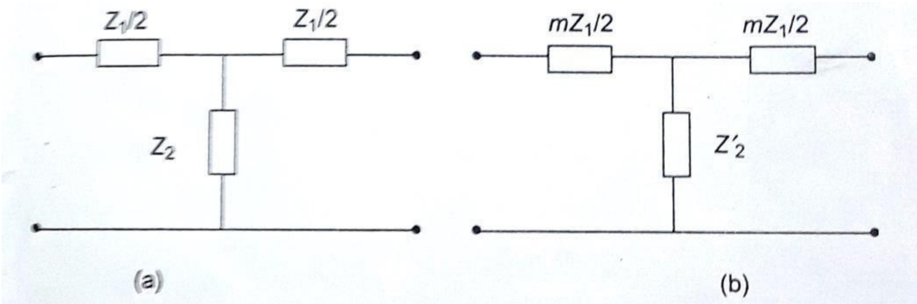


Fig: (a)A constant ‘k’ type ‘T’ section (b) a ‘m’ derived filter with a suitably modified ‘T’ section

Our Objective is to find the value od Z_2’ so as to maintain the characteristic impedances of both the T sections Identical. It is obtained in the following steps.

The characteristic Impedance of a normal constant ‘k’ type filter with a ‘T’ section is given by:

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

Similarly the characteristic Impedance of a normal constant ‘m’ derived filter with a ‘T’ section is given by:

$$Z_{0T'} = \sqrt{\frac{m^2 Z_1^2}{4} + m Z_1 Z_2'}$$

Since the m’ derived filter is designed to have the same characteristic Impedance as the constant ‘k’ type filter

$$Z_{0T} = Z_{0T'} \qquad \text{i.e.} \qquad \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{\frac{m^2 Z_1^2}{4} + m Z_1 Z_2'}$$

The value of Z_2' is obtained by solving the above equation for Z_2' as below:

Squaring the above equation on both sides and rearranging gives :

$$\frac{Z_1^2}{4} + Z_1 Z_2 = \frac{m^2 Z_1^2}{4} + m Z_1 Z_2'$$

$$m Z_1 Z_2' = \frac{Z_1^2}{4}(1 - m^2) + Z_1 Z_2$$

$$Z_2' = \frac{Z_1}{4m}(1 - m^2) + \frac{Z_2}{m}$$

The equivalent 'T' section of the 'm' derived filter using the above value of Z_2' is shown in the figure below.

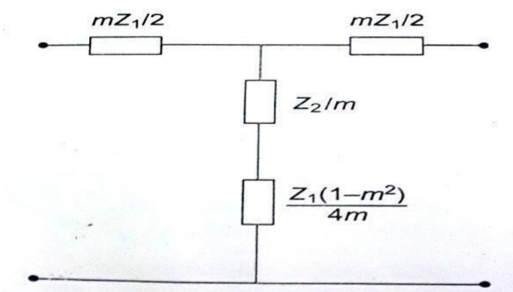


Fig: Equivalent T section of the 'm' derived filter

It can be seen from this figure that Z_2' consists of two impedances Z_2/m and $Z_1(1-m^2)/4m$ connected in series. From the above expression for Z_2' it can be seen that $(1-m^2)/4m$ should be positive to realize the impedance Z_2' physically. i.e. $0 < m < 1$. Thus it could be seen that a 'm' derived filter could be obtained from a simple constant 'k' type filter by modifying its series and shunt arms as shown in the figure above and with values of Z_1' & Z_2' given as below .

$Z_1' = m Z_1/2$ and $Z_2' = Z_2/m + Z_1(1-m_2)/4m$

'm'derived π section:

The same method as was used for the 'T' section can be used here to get the equivalent 'm' derived filter with a π section. A constant 'k' type π section and a 'm' derived filter with a suitably modified 'π' section configuration are shown in the figure below.

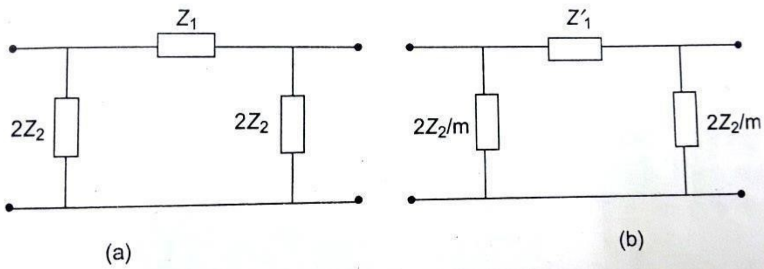


Fig: A constant 'k' type π section and a 'm' derived filter with a suitably modified 'π' section

Our Objective is to find the value od Z_1' so as to maintain the characteristic impedances of both the **π** *sections Identical*. It is obtained in the following steps.

The characteristic Impedance of a normal constant 'k' type filter with a 'π' section is given by:

$$Z_{0\pi} = \sqrt{\frac{Z_1 Z_2}{1+\frac{Z_1}{4Z_2}}}$$

Similarly the characteristic Impedance of a normal constant 'm' derived filter with a 'π' section is given by:

$$Z'_{0\pi} = \sqrt{\frac{Z'_1 \frac{Z_2}{m}}{1+\frac{Z'_1}{4\cdot Z_2 / m}}}$$

Since the 'm' derived filter is designed to have the same characteristic Impedance as the constant 'k' type filter:

$$Z_{0\pi} = Z'_{0\pi} \qquad \therefore \qquad \sqrt{\frac{Z_1 Z_2}{1+\frac{Z_1}{4Z_2}}} = \sqrt{\frac{Z_1' \frac{Z_2}{m}}{1+\frac{Z_1'}{4\cdot Z_2 / m}}}$$

The value of Z_1' is obtained by solving the above equation for Z_1' as below:

Squaring and cross multiplying both the sides of the above equation gives:

$$(4Z_1Z_2 + mZ_1'Z_1) = \frac{4Z_1'Z_2 + Z_1Z_1'}{m}$$

$$Z_1'\left(\frac{Z_1}{m} + \frac{4Z_2}{m} - mZ_1\right) = 4Z_1Z_2$$

$$Z_1' = \frac{Z_1Z_2}{\frac{Z_1}{4m} + \frac{Z_2}{m} - \frac{mZ_1}{4}}$$

$$= \frac{Z_1Z_2}{\frac{Z_2}{m} + \frac{Z_1}{4m}(1-m^2)}$$

$$Z_1' = \frac{Z_1Z_2\frac{4m^2}{(1-m^2)}}{\frac{Z_2 4m^2}{m(1-m^2)} + Z_1 m} = \frac{mZ_1\frac{Z_2 4m}{(1-m^2)}}{mZ_1 + \frac{Z_2 4m}{(1-m^2)}}$$

The equivalent π section of the ‘m’ derived filter using the above value of Z_1' is shown in the figure below.

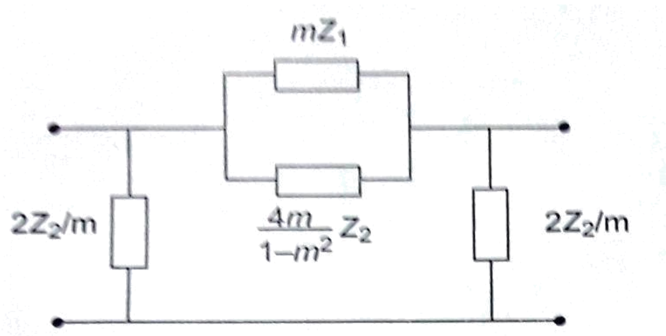


Fig: Equivalent π section of the 'm' derived filter

It can be seen from this figure that Z_1' consists of two impedances mZ_1 and $[4m/(1-m^2)]\,Z_2$ connected in parallel. Thus it could be seen that a 'm' derived filter could be obtained from a simple constant 'k' type filter by modifying its series and shunt arms as shown in the figure above.

'm' derived LPF:

Analysis:

'm' derived low pass filter using L and C with T and π sections is shown in the figure below.

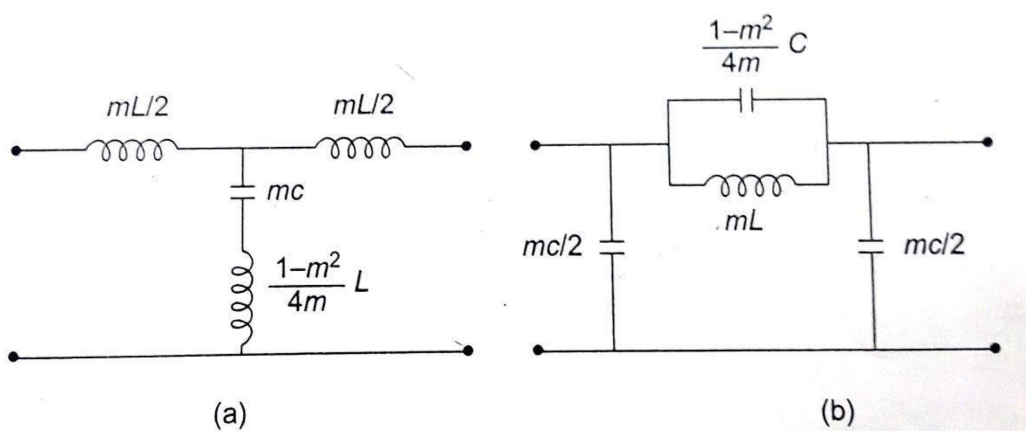


Fig: 'm' derived low pass filter with L and C in T and π sections

T section:

In the T section the tuned circuit is in *Series connection* and is in the *shunt arm*. At resonance a series tuned circuit offers minimum resistance and since it is in the shunt arm the filter attenuates the input signal completely at the resonance frequency. The resonant frequency is denoted by f_∞

At resonance frequency the inductive reactance is equal to the capacitive reactance.

i.e. $\omega_\infty L[(1-m^2)/4m] = 1/\omega_\infty C.m$

Hence the resonance frequency is given by :

But we know that the cutoff frequency f_c is given by $f_c = 1/\pi\sqrt{LC}$ and hence $\omega_c^2 = 4/LC$ Dividing the expression for ω_∞^2 with expression for ω_c^2 we get

$$\frac{\omega_\infty^2}{\omega_C^2} = \frac{4}{LC(1-m^2)} \times \frac{LC}{4} = \frac{1}{1-m^2}$$

$$\omega_\infty = \frac{\omega_c}{\sqrt{1-m^2}}$$

And therefore $$f_\infty = \frac{f_c}{\sqrt{1-m^2}}$$

Which gives $$m = \sqrt{1-\frac{f_c^2}{f_\infty^2}}$$

It has to be observed that in the case of a LPF the cutoff frequency f_c is smaller than the resonant frequency f_∞ and thus 'm' satisfies the basic design condition 0 <'m' < 1

π section:

In the 'π' section the tuned circuit is in *parallel connection* and is in the *series arm*. At resonance the parallel tuned circuit offers the maximum resistance and since it is now in series arm it offers maximum attenuation. Hence in a LPF either with 'T' section or 'π' section the resonance frequency is same and at resonance both work as Low Pass Filters.

Hence for a π section LPF f_∞ is given by the same value $$f_\infty = \frac{f_c}{\sqrt{1-m^2}}$$ and the value of 'm' is

also given by the same value $$m = \sqrt{1 - \frac{f_c^2}{f_\infty^2}}$$ and again satisfies the design condition *0 <'m' < 1*

'm' derived HPF:

Analysis:

'm' derived high pass filter with L and C in 'T' and 'π' sections is shown in the figure below.

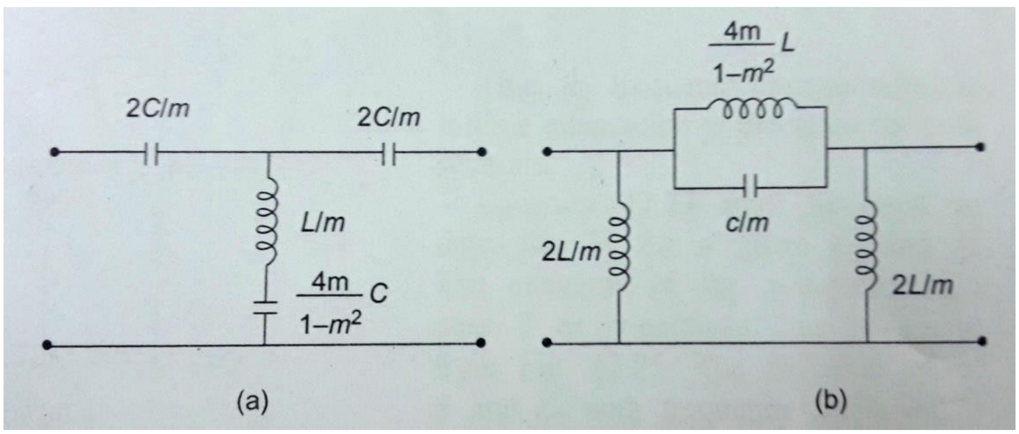


Fig: 'm' derived high pass filter with L and C in 'T' and 'π' sections

T section:

In the T section the tuned circuit is in Series connection and is in the shunt arm. It consists of an Inductance (L/m) in series with a capacitance C. [4m /(1-m^2)] At resonance a series tuned circuit offers minimum resistance and since it is in the shunt arm the filter attenuates the input signal completely at the resonance frequency. The resonant frequency is denoted by f_∞

At resonance frequency the inductive reactance is equal to the capacitive reactance.

$$\frac{1}{\frac{4m}{1-m^2}\cdot\omega_\infty\cdot C}=\frac{\omega_\infty L}{m}$$

i.e.

Hence the resonance frequency is given by :

$$\omega_\infty^2=\frac{1-m^2}{4LC}$$

i.e.

$$\omega_\infty=\frac{\sqrt{1-m^2}}{2\sqrt{LC}}$$

i.e.

$$f_\infty=\frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$

But we know that the cutoff frequency f_c for a HPF is given by $f_c = 1/4\pi\sqrt{LC}$ hz and using this relation in

$$f_\infty=(\sqrt{1-m^2})\,f_c$$

the above expression for f_∞ we get

$$m=\sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}=\sqrt{1-\left(\frac{\omega_\infty}{\omega_C}\right)^2}$$

Which gives

It has to be observed that in the case of a HPF the cutoff frequency f_c is larger than the resonant frequency f_∞ and thus 'm' satisfies the basic design condition 0 <'m' < 1

π section:

In the 'π' section the tuned circuit is in *parallel connection* and is in the *series arm*. At resonance the parallel tuned circuit offers the maximum resistance and since it is now in series arm it offers maximum attenuation. Hence in a LPF either with T section or π section the resonance frequency is same and at resonance both work as high Pass Filters.

Hence for a ‘π’ f_∞ is given by the same value

$$f_\infty = \left(\sqrt{1-m^2}\right) f_c$$

and the value of ‘m’ is also given by the same value

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} = \sqrt{1-\left(\frac{\omega_\infty}{\omega_C}\right)^2}$$

and again satisfies the design condition *0 <‘m’ < 1.*

BAND PASS FILTER:

A band pass filter can be thought of as a set of Low pass filter and high pass filter connected in tandem with the cutoff frequencies designed such that the HPF cutoff frequency is lower than the cutoff frequency of the LPF. Then the HPF cutoff frequency becomes the Lower cutoff frequency of the BPF and the LPF cutoff frequency becomes the upper cutoff frequency of the BPF. A band pass filter in both T and π configurations is shown in the figure below. ***It is a combination of one (or two) series tuned circuit/s placed in series Path and one (or two) Parallel tuned circuit/s placed in the shunt path***. The tuned circuits will have a resonant frequency***. At resonance frequency, a series tuned circuit offers lowest resistance and a parallel tuned circuit offers maximum resistance.*** Thus the circuit allows passage of signals within the pass band of frequencies around the resonant frequency and stops passage of all other signals on either side of the pass band.

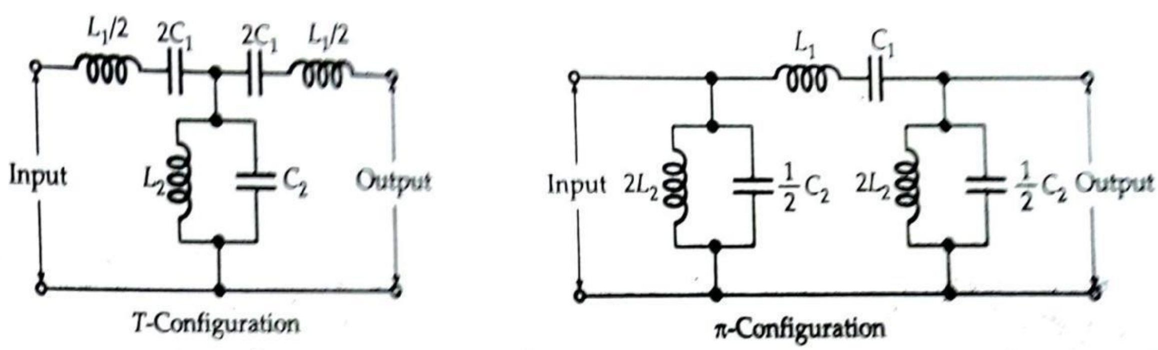


Fig: Band Pass filter In T and π configurations

The design of a band pass filter involves the selection of the values of L1,C1, L2 and C2 given the required band of frequencies i.e lower cutoff frequency f1 (Hz) ,the upper cutoff frequency f2 (Hz)and the design resistance R0(ohms). These design expressions are given below. Once these values are determined, the circuit as configured above can be realized in either T or π configurations.

$L_1 = R_0/\pi\,(f_2-f_1)$ Henrys

$L_2 = R_0\,(f_2-f_1)/4\pi f_1 f_2$ Henrys

$C_1 = (f_2-f_1)/4\pi f_1 f_2 R_0$ Farads

$C_2 = 1/\pi R_0\,(f_2-f_1)$ Farads

BAND ELIMINATION (STOP) FILTER:

A band stop filter can be thought of as a set of Low pass filter and high pass filter connected in tandem with the cutoff frequencies designed such that the HPF cutoff frequency is higher than the cutoff frequency of the LPF. Then the LPF cutoff frequency becomes the Lower cutoff frequency of the BSF and the HPF cutoff frequency becomes the upper cutoff frequency of the BSF. A band stop filter in both T and π configurations is shown in the figure below**.** *It is a combination of one (or two) series tuned circuit/s placed in **Shunt Path** and one (or two) Parallel tuned circuit/s placed **in the series path***. The tuned circuits will have a resonant frequency. At resonance frequency, a series tuned circuit offers lowest resistance and a parallel tuned circuit offers maximum resistance. Thus the circuit does not allow passage of signals within the stop band of frequencies around the resonant frequency and allows passage of all other signals on either side of the stop band.

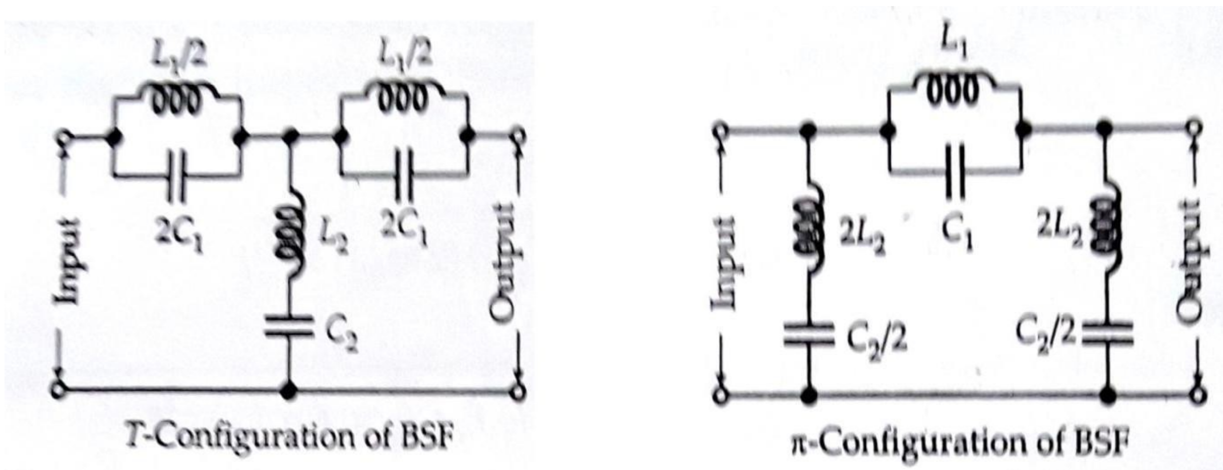


Fig: Band Stop Filter In T and π configurations

The design of a Band Stop Filter involves the selection of the values of $\mathbf{L_1, C_1, L_2}$ and $\mathbf{C_2}$, given the required band of frequencies i.e. lower cutoff frequency **f_1 (Hz)** ,the upper cutoff frequency **f_2 (Hz)**and the design resistance **R_0(ohms).** These design expressions are given below. Once these values are determined, the circuit as configured above can be realized in either **T** or **π** configurations.

$L_1 = R_0\,(f_2-f_1)/\pi f_1 f_2$

$C_1 = 1/4\pi R_0\,(f_2-f_1)$

$L_2 = R_0/4\pi^2(f_2-f_1)$

$C_2 = (f_2-f_1)/\pi f_1 f_2 R_0$

SYMMETRICAL ATTENUATORS:

Symmetrical Two port Networks are defined as networks in which the Input and output ports can be interchanged without changing the input/output Voltages/currents. Hence Symmetrical attenuators are designed such that both the pairs of terminals are matched to the same characteristic Resistance R_0.This is achieved by keeping both the input side and output side resistors same. The attenuation is measured in nepers or decibels. The decibel is defined as:

DB = 20 $\log_{10} I_1/I_2$. Defining $N = I_1/I_2$ then

DB = 20 $\log_{10}$ N and

N = Antilog DB/20

Normally Attenuators re required to be designed with a given level of attenuation in Decibels. So first the given level of attenuation in decibels is to be converted from decibels into N. There are different types of attenuators and we will explain the design of the following four important types of Attenuators.

The design of the Attenuators involves finding of the Values of the resistors for the given type of configuration when the required level of attenuation either in DB or as a ratio of I_1 & I_2 i.e N and the Characteristic Impedance (Resistance R_0) are specified.

Symmetrical T attenuator:

The configuration of a symmetrical T type Attenuator terminated in its characteristic Resistance R_0 is shown in the figure below. As can be seen the attenuator is symmetrical with both the input side and output side resistors being the same. (R_1 and R_1)

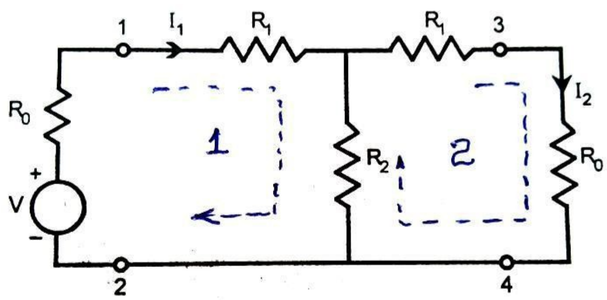


Fig: Symmetrical T type Attenuator

Writing the current equation for loop 2

$$I_2 (R_1 + R_2 + R_0) - I_1 R_2 = 0$$

$$\frac{I_1}{I_2} = \frac{R_1 + R_2 + R_0}{R_2}$$

But $\frac{I_1}{I_2} = N$

Therefore $N = \frac{R_1 + R_2 + R_0}{R_2}$

We know that the impedance looking into the terminals 1 &2 when the network is terminated in its Resistance R_0 is equal to R_0 itself and therefore

$$R_0 = R_1 + \frac{R_2(R_0 + R_1)}{R_1 + R_2 + R_0}$$

Using the above expression for N into the above expression for R_0 we get

$$R_0 = R_1 + \frac{R_0 + R_1}{N}$$

Solving for R_1 we get

$$R_1 = \frac{R_0(N-1)}{N+1}$$

Substituting the above value of R_1 into the expression for N and solving for R_2we get

$$R_2 = \frac{2R_0 N}{N^2 - 1}$$

Symmetrical π attenuator:

The configuration of a symmetrical π type Attenuator terminated in its characteristic Resistance R_0 is shown in the figure below. As can be seen the attenuator is symmetrical with both the input side and output side resistors being the same. (R_2 and R_2)

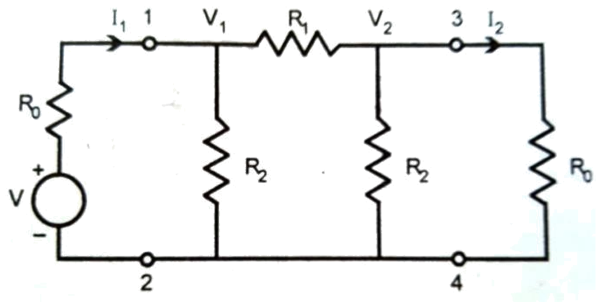


Fig: Symmetrical π type Attenuator

Writing the equation for current at Node 3 we get

$$V_2\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_0}\right)-\frac{V_1}{R_1}=0$$

Solving the above equation for V_1/V_2 we get

$$\frac{V_1}{V_2}=R_1\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_0}\right]$$

But $V_1/V_2 = I_1/I_2 = N$

$$\therefore \quad N=\frac{R_0R_2+R_1R_0+R_1R_2}{R_2R_0}$$

We know that the impedance looking into the terminals 1 &2 when the network is terminated in its Resistance R_0 is equal to R_0 itself and therefore

$$R_0=\frac{R_2\left[R_1+\frac{R_0R_2}{R_0+R_2}\right]}{R_2+R_1+\frac{R_0R_2}{R_0+R_2}}$$

$$=\frac{R_2[R_1R_0+R_1R_2+R_0R_2]}{R_2(R_0+R_2)+(R_1R_0+R_1R_2+R_0R_2)}$$

Using the above final expression for N into the above expression for R_0 we get

$$R_0 = \frac{R_2^2 R_0 N}{R_2(R_0 + R_2) + NR_2R_0}$$

Solving for R_2 we get

$$R_2 = R_0\left(\frac{N+1}{N-1}\right)$$

Substituting the above value of R_2 in to the expression for N and solving for R_1 we get

$$R_1 = \frac{R_0(N^2-1)}{2N}$$

Symmetrical Bridged T attenuator:

The next symmetrical attenuator which is commonly used is a Symmetrical Bridged T attenuator.The configuration of a symmetrical π type Attenuator terminated in its characteristic Resistance R_0 is shown in the figure below. As can be seen the attenuator is symmetrical with both the input side and output side resistors being the same. (R_1 and R_1)

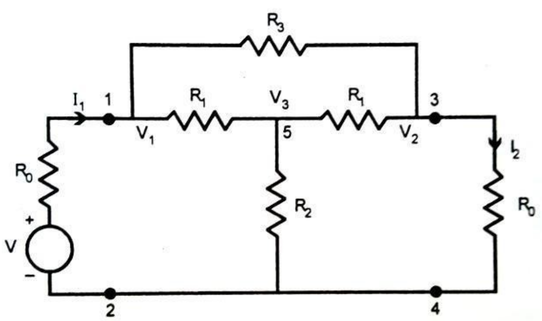


Fig: Symmetrical Bridged T attenuator

First we have to get an expression for the characteristic resistance R_0 in terms of the Network resistances R_1, R2 and R_3

Since the Bridged T type Attenuator shown in the figure above is symmetrical Network, the characteristic Resistance R_0 is given by

$$R_0 = \sqrt{R_{SC}R_{OC}}$$

R_{SC} and R_{OC} are the short circuit resistance and open circuit resistance of the Bridged T type Attenuator as seen from the terminals 1and 2 shown in the figure above. Thus

$$R_{SC} = \frac{R_3\left(R_1 + \dfrac{R_1R_2}{R_1 + R_2}\right)}{R_3 + R_1 + \dfrac{R_1R_2}{R_1 + R_2}}$$

$$= \frac{R_3\left(2R_1R_2 + R_1^2\right)}{R_3R_1 + R_3R_2 + R_1^2 + 2R_1R_2}$$

$$R_{OC} = R_2 + \frac{R_1(R_1 + R_3)}{2R_1 + R_3}$$

$$= \frac{2R_1R_2 + R_2R_3 + R_1^2 + R_1R_3}{2R_1 + R_3}$$

And

$$R_0 = \sqrt{\frac{R_3\left(2R_1R_2 + R_1^2\right)}{2R_1 + R_3}}$$

Usually in the design of a Bridged T type Attenuator R_1 is chosen to be equal to R_0 and the resistances R_2 and R_3 only are made variable and are chosen so as to get the specified values of R_0 and N. Hence substituting R_0 in place of R_1 into the above equation for R_0 we get

$$R_0 = \sqrt{\frac{R_3\left(2R_0R_2 + R_0^2\right)}{2R_0 + R_3}}$$

$$R_0^2 = \frac{2R_0R_2R_3 + R_0^2R_3^{\cancel{2}}}{2R_0 + R_3}$$

Solving the above equation for R_0 we get

$$R_0 = \sqrt{R_2 R_3}$$

Referring to the figure above and writing the equations for current at nodes 3 &5 and setting $R_1 = R_0$ we get

At Node 3

$$V_2\left(\frac{1}{R_0}+\frac{1}{R_0}+\frac{1}{R_3}\right)-\frac{V_1}{R_3}-\frac{V_3}{R_0}=0$$

And at Node 5

$$V_3\left(\frac{2}{R_0}+\frac{1}{R_2}\right)-\frac{V_2}{R_0}-\frac{V_1}{R_0}=0$$

Eliminating V3 from the above two equations we get

$$\frac{V_1}{V_2}=\frac{\dfrac{2R_3+R_0}{R_0R_3}-\dfrac{R_2}{R_0(2R_2+R_0)}}{\dfrac{1}{R_3}+\dfrac{R_2}{R_0(2R_2+R_0)}}$$

But $V_1/V_2 = I_1/I_2 = N$ and so we get

$$N=\frac{(2R_2+R_0)(2R_3+R_0)-R_2R_3}{2R_0R_2+R_0^2+R_2R_3}$$

Substituting $R_0^2 = R_2R_3$ and simplifying we get

$$N = 1 + \frac{R_3}{R_0}$$

Since $R_0^2 = R_2R_3$ the above equation can also be written as

$$N = 1 + \frac{R_0}{R_2}$$

And from the above two equations we can get the values of R_3 and R_2 as

$$R_3 = R_0 (N-1) \text{ and}$$

$$R_2 = R_0 / (N-1)$$

A Bridged T type Attenuator is more economical and convenient for design and use as only two resistances have to be changed /varied to realize a particular value of R_0 and N where as in 'T' and 'π' type attenuators 3 resistances have to be changed.

Symmetrical Lattice attenuator:

The configuration of a Symmetrical Lattice attenuator is shown in the figure (a) below. For ease of analysis and understanding it is redrawn and shown as Bridge Network in figure (b)

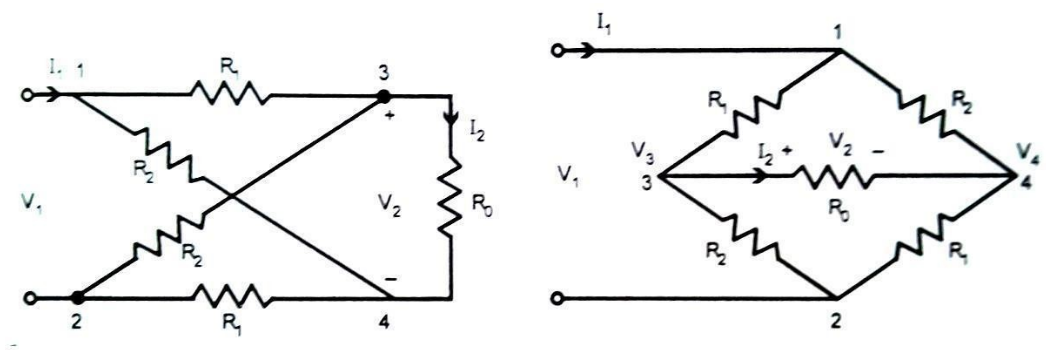


Fig: (a) Symmetrical Lattice attenuator (b) Redrawn as a Bridge Network

Since the lattice Attenuator shown in the figure above is also a symmetrical Network we can use the

same relation $R_0 = \sqrt{R_{SC} R_{OC}}$ to get the values of R_1 and R_2 .

From figure (b) above we have

$$Z_{SC} = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_1 R_2}{R_1 + R_2}$$

$$= \frac{2R_1 R_2}{R_1 + R_2}$$

$$Z_{OC} = \frac{(R_1 + R_2)}{2}$$

But we know that $$R_0 = \sqrt{Z_{SC}.Z_{OC}}$$

Substituting the above values values of Z_{SC} and Z_{OC} in theabove expression for R_0 we get

$$R_0 = \sqrt{Z_{SC}.Z_{OC}} \quad = \sqrt{\frac{2R_1R_2}{R_1 + R_2}.\frac{(R_1 + R_2)}{2}} = \sqrt{R_1R_2}$$

Referring to the figure above and writing the equations for current at nodes 3 & 4 taking node 2 as reference we have at node 3:

$$\frac{V_3 - V_4}{R_0} + \frac{V_3 - V_1}{R_1} + \frac{V_3}{R_2} = 0$$

And at node 4:

Noting that $V_3 - V_4 = V_2$ the above two equations will become

$$\frac{V_4 - V_3}{R_0} + \frac{V_4}{R_1} + \frac{V_4 - V_1}{R_2} = 0$$

$$\frac{V_2}{R_0} + V_3\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_1}{R_1}$$

$$-\frac{V_2}{R_0} + V_4\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_1}{R_2}$$

Subtracting the second equation from the first equation and again replacing (V_3-V_4) by V_2 we get

$$\frac{2V_2}{R_0} + V_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_1\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Or

$$V_2\left[\frac{2R_1R_2 + R_0R_1 + R_2R_0}{R_0R_1R_2}\right] = V_1\left[\frac{R_2 - R_1}{R_1R_2}\right]$$

Thus

$$\frac{V_1}{V_2} = N = \frac{1}{R_0} \cdot \frac{(2R_1R_2 + R_0R_1 + R_2R_0)}{R_2 - R_1}$$

And using the already established relation $R_0^2 = R_1R_2$ in the above expression for N we get

$$N = \frac{2\sqrt{R_1R_2} + R_1 + R_2}{R_2 - R_1} = \frac{\left(\sqrt{R_1} + \sqrt{R_2}\right)^2}{\left(\sqrt{R_2} - \sqrt{R_1}\right)\left(\sqrt{R_2} + \sqrt{R_1}\right)}$$

$$N = \frac{1 + \sqrt{\frac{R_1}{R_2}}}{1 - \sqrt{\frac{R_1}{R_2}}}$$

Or in terms of R_0 and R_1

$$N = \frac{1 + \frac{R_1}{R_0}}{1 - \frac{R_1}{R_0}}$$

From which we get

$$R_1 = R_0\left(\frac{N-1}{N+1}\right)$$

And

$$R_2 = R_0\frac{(N+1)}{N-1}$$

Important concepts and formulae :

Characteristic Impedance of a T Network :

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

Characteristic Impedance of a π Network :

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2 + Z_1^2 / 4}}$$

Relationship between the characteristic Impedances of T and π networks :

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}}$$

Constant K type LPF:

The following points are to be noted for drawing the configuration :

- *The L and C values are same for both T and π configurations.*
- *For a LPF the inductance comes in the series arm (Inductance offers low impedance at low frequencies and low impedance is required in Series arm for passage of signal in the required band).It comes as one element L in the π Configuration and as two elements each of L/2 in the T Configuration.*
- *For a LPF the capacitance comes in the shunt arm (capacitance offers high impedance at low frequencies and high impedance is required in Shunt arm for passage of signal in the required band). It comes as one element C in the T Configuration and as two elements each of C/2 in the π Configuration.*
- *Then using the following formulae for L and C the final circuit diagram can be drawn*

$$R_0 = \sqrt{\frac{L}{C}} \qquad L = \frac{R_0}{\pi f_c}$$

$$f_C = \frac{1}{\pi\sqrt{LC}} \qquad C = \frac{1}{\pi R_0 \, f_C}$$

Constant K type HPF :

The following points are to be noted here while drawing the configuration:

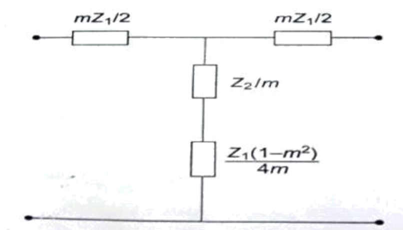
- *The L and C values are same for both T and π configurations.*
- *For a HPF the capacitance comes in the series arm (capacitance offers low impedance at high frequencies and low impedance is required in Series arm for passage of signal in the required band)). It comes as C when it is one element as in π configuration and as 2C when it is divided as two in series in the T configuration.*
- *For a HPF the inductance comes in the shunt arm (inductance offers high impedance at high frequencies and high impedance is required in Shunt arm for passage of signal in the required band)). It comes as L when it is one element as in T configuration and comes as 2L when it is divided into two elements in parallel as in π configuration.*
- *Then using the following formulae for L and C the final circuit diagram can be drawn*

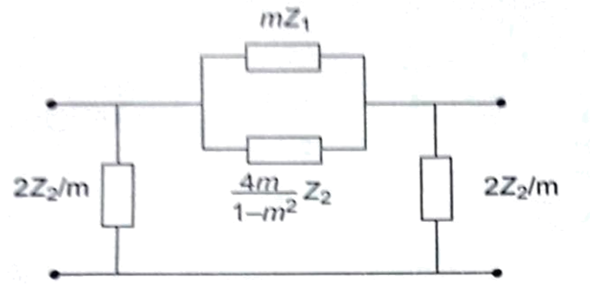
$$R_0 = \sqrt{\frac{L}{C}} \qquad f_C = \frac{1}{4\pi\sqrt{LC}} \qquad L = \frac{R_0}{4\pi f_C} \qquad C = \frac{1}{4\pi R_0 f_C}$$

‘m’ derived filter

A ‘m’ derived filter is identical to the constant ‘k’ type filter except that

- In a T section the series arm impedance is ***multiplied*** by the constant **‘m’** and
- In a π section the shunt arm impedance is ***divided*** by the constant **‘m’**
- Then the other arm resistances are derived so as to maintain the characteristic impedance same. The net impedance values are shown in the figure below .





T configuration of m derived filter π configuration of m derived filter

- For a m derived LPF :

$$m=\sqrt{1-\frac{f_c^2}{f_\infty^2}}$$

- For a m derived High Pass Filter :

$$m=\sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}=\sqrt{1-\left(\frac{\omega_\infty}{\omega_C}\right)^2}$$

- Using the above points and formulae along with the rules mentioned in the normal K type filters m derived filters with L and C can be designed can be designed.

$C_1 = (f_2-f_1) / 4\pi f_1f_2R_0$ $C_2 = 1 / \pi R_0 (f_2-f_1)$

Band pass filter

H
e
nr
ys

Farads

Farads

Henrys

The following points are to be noted here while drawing the configuration:

- *The L and C values are same for both T and π configurations.*
- *For a BPF the series tuned circuit with L and C comes in the series arm (series*

tuned circuit offers low impedance at the resonant frequency and low impedance is required in Series arm for passage of signal in the required band *around the resonant frequency). It comes as C when it is one element as in π configuration and as 2C when it is divided as two in series in the T configuration.*

- *For a BPF the shunt tuned circuit with L and C comes in the shunt arm (shunt tuned circuit with L and C offers high impedance at resonant frequency and high impedance is required in Shunt arm for passage of signal around the resonant frequency). It comes as L when it is one element as in T configuration and comes as 2L when it is divided into two elements in parallel as in π configuration*

Band Stop filter

$= (f_2 - f_1)/\pi f_1 f_2$

R_0

L_1 Henrys Farads

$L_2 = R_0/4\pi(f_2 - f_1)$

$C_2 = (f_2 - f_1)/\pi f_1 f_2 R_0$

Henrys Farads

The following points are to be noted here while drawing the configuration:

- *The L and C values are same for both T and π configurations.*
- *For a BSF the shunt tuned circuit with L and C comes in the series arm (shunt tuned circuit offers maximum impedance at the resonant frequency and*

  maximum impedance is required in Series arm for stopping of signal in the required band around the resonant frequency).
- *For a BSF the series tuned circuit with L and C comes in the shunt arm (series tuned circuit with L and C offers minimum impedance at resonant frequency and low impedance is required in Shunt arm for stopping of signal around the resonant frequency).*

Attenuators:

- **Symmetrical T type attenuators:**

$$R_1 = \frac{R_0(N-1)}{N+1} \quad \text{and} \quad R_2 = \frac{2R_0N}{N^2-1}$$

- **Symmetrical π type Attenuators:**

and

- **Symmetrical Bridged T network attenuators:**

$R_2 = R_0 / (N-1)$

$R_3 = R_0 (N-1)$ and

Symmetrical Lattice Attenuators:

$$R_1 = R_0\left(\frac{N-1}{N+1}\right) \quad \text{And} \quad R_2 = R_0\frac{(N+1)}{N-1}$$

Previous year Question papers:

R09 May11

1. Explain *T – type attenuator* and also design a T – type attenuator to give an attenuation of 60dB and to work in a line of 500Ω impedance. (R09 May 11)
2. Design a m – derived high pass filter with a cut – off frequency of 10KHz; design impedance of 5Ω and m = 0.4. (R09 May 11)
3. Explain the lattice attenuator and also design a lattice attenuator to have a characteristic impedance of 800Ω and attenuation of 20 dB. (R09 May 11)
4. What is a constant – K low pass filter, derive its characteristics impedance. (R09 May 11)
5. Explain *π – type attenuator* and also design it to give 20db attenuation and to have characteristic impedance of 100Ω. (R09 May 11)
6. A low pass π section filter consists of an inductance of 25 mH in series arm and two capacitors of 0.2μF in shunt arms. Calculate the cut – off frequency, design impedance, attenuation at 5 KHz and phase shift at 2 KHz. Also find the characteristic impedance at 2 KHz. (R09 May 11)
7. Explain *Bridged – T attenuator* and also design it with an attenuation of 20 dB and terminated in a load of 500Ω. (R09 May 11)

R09 May 12

8. Explain m-derived low-pass T-section and π section in detail and the necessary Design procedure.
9. An attenuator is composed of symmetrical T-section having series arm each of 420 ohms and shunt arm of 740 ohms. Derive expression for and calculate the characteristic impedance of this network and attenuation per section. Draw the circuit diagram for symmetrical T-type attenuator.
10. (a) Explain *symmetrical π type attenuator* with necessary equations in detail.

(b) Design a symmetrical π type attenuator to give 20 db attenuation and to have a characteristic impedance of 200 ohms.Derive the expression for symmetrical T and π filter networks.

11. What is an Attenuator? Explain diferent types of symmetrical attenuators indetail?
12. (a) Derive the necessary expressions for m-derived low pass filter.
    (b) Derive the necessary expression for m-derived high pass filter.

13. (a) Design a T-type attenuator to have an attenuation of 40db and to work between source impedance of 400 ohms and load impedance of 900 ohms.
    (b) Design a _-type attenuator to have an attenuation of 25db and to work between Source impedance of 600 ohms and load impedance of 1000 ohms.

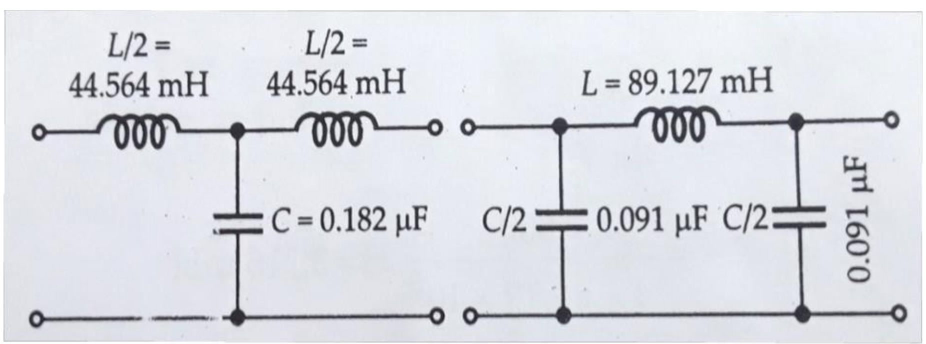
Illustrative Examples:

Ex.1: Design a constant K type low pass filter having a cutoff frequency of 2.5 kHz and a design resistance (Impedance) of 700 Ω in both T and π configurations.

Solution: From the given data we have

f_c = 2.5 khz = 2500 Hz and R_0 = 700 Ω

The basic configuration of a LPF in T and π configurations is shown below.



(a) T configuration LPF (b) π configuration LPF

For a LPF we have the design values of L and C in terms of f_c and R_0 as :

$L = R_0 / \pi . f_c$ and

$C = 1/\pi . R_0 . f_c$

and using them we get :

$L = R_0 / \pi . f_c$ = 700/ π.2500 Henrys = 89.127 mH ; L/2 = 44.56 mH and

$C = 1/\pi.R_0$
$.f_c$ $= 1/\pi x700x2500$ Farads = 0.182 μF ; C/2 = 0.091 μF

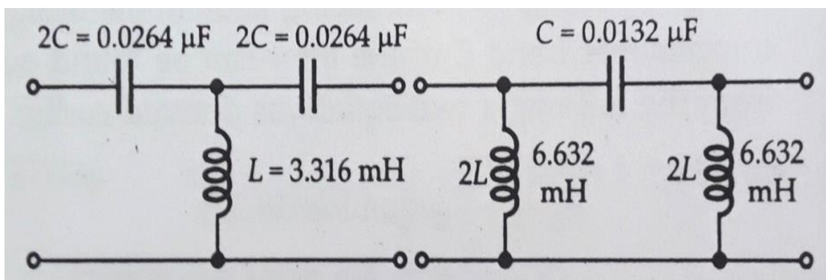
These values are shown substituted in the figure above.

Ex.2: Design a proto type (it is same as constant K type) high pass filter having a cutoff frequency of 12 kHz and a design resistance (Impedance) of 500 Ω in both T and π configurations.

Solution: From the given data we have

f_c = 12 khz = 12000 Hz and R_0 = 500 Ω

The basic configuration of a HPF in T and π configurations is shown below:



(a) T configuration HPF **(b) π Configuration HPF**

For a HPF (High Pass Filter) we have the design values of L and C in terms of f_c and R_0 as :

L = R_0 / 4π. f_c and

C = 1/4π.R_0 .f_c

and using them we get :

L = R_0 / 4π. f_c	= 500/ 4π.12000 Henrys	= 3.316 mH	; 2L = 6.632 mH and
C = 1/4π.R_0 .f_c	= 1/4 πx500x12000 Farads	= 0.0132 μF	; 2C = 0.0264 μF

These values are shown substituted in the figure above:

Ex.3: Design a proto type Band Pass Filter in both T and π configurations having cutoff frequencies of 3000 Hz and 6000 Hz and Nominal Characteristic Impedance of 600 Ω.

Solution: From the given data we have :

f_2 = 6000 Hz, f_1 = 3000 Hz and R_0 = 600 Ω

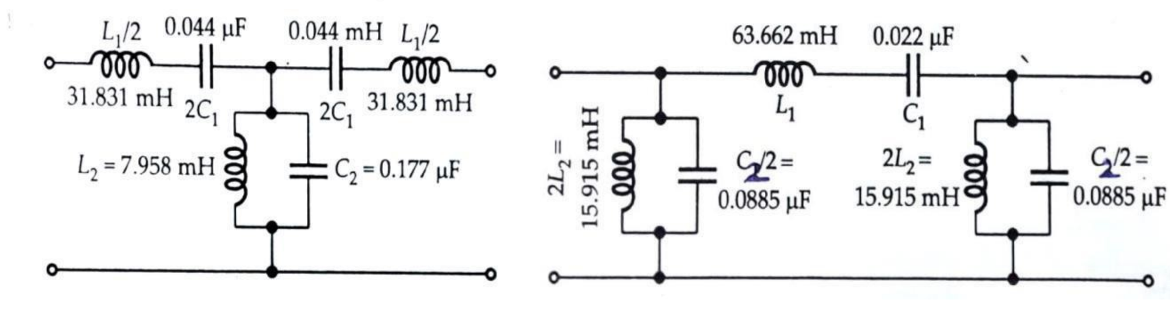


Fig: (a) BPF in T configuration (b) BPF in π configuration

For a BPF (Band Pass Filter) we have the design values of L_1, C_1, L_2 and C_2 in terms of f_1, f_2 and R_0 as:

$L_1 = R_0 / \pi (f_2 – f_1)$ henrys

$L_2 = R_0 . (f_2 – f_1) / 4 \pi f_1 f_2$ henrys

$C_1 = (f_2 – f_1) / 4 \pi R_0 f_1 f_2$ Farads

$C_2 = 1 / \pi R_0 (f_2 - f_1)$ Farads

And using them we get :

Series arm inductance **L_1** = $R_0 / \pi (f_2 – f_1)$ henrys = 600/ π (6000 – 3000) H = **63.662 mH**

and **$L_1/2$** = 63.662/2 = **31.83 mH**

Shunt arm inductance **L_2** = $R_0 (f_2 – f_1) / 4 \pi f_1 f_2$ henrys = **7.96 mH**

= 600(6000 – 3000)/ 4 π x 3000 x 6000 H

and 2**L_2** = **2 x 7.96 mH = 15.92 mH**

Next :

Series arm capacitance $\mathbf{C_1}$ = $(f_2 - f_1) / 4 \pi R_0 f_1 f_2$ Farads

$= (6000 – 3000)/ 4 \pi \times 600 \times 3000 \times 6000$ F = **0.022 µF**

and **$2C_1$** $= 2 \times 0.022$ **= 0.044 µF**

Shunt arm capacitance **C_2** $= 1 / \pi R_0 (f_2 – f_1)$ Farads

$= 1 / \pi \times 600 \times (6000 – 3000)$ F = **0.177 µF**

and **$C_2 /2$** = 0.177/2 **= 0.0885 µF**

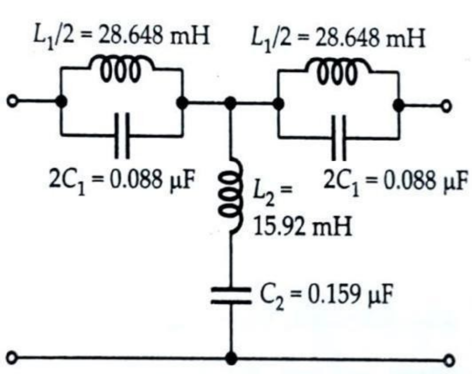
Next:
The resonant frequency **f_0** of the **BPF** is given by **$f_0 = \sqrt{f_1 f_2}$** $= \sqrt{3000 \times 6000}$ = **4242.640 Hz**

Ex.4: Design a constant K type Band Stop Filter in both T and π configurations having cutoff frequencies of 2000 Hz & 5000 Hz and design Resistance of 600 Ω.

Solution: From the given data we have :

$f_2 = 5000$ Hz, $f_1 = 2000$ Hz and $R_0 = 600\ \Omega$

The basic configuration of a BSF in both T and π configurations is shown in the figure below.



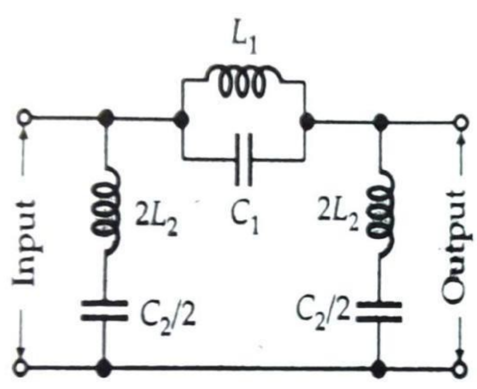


Fig: (a) BSF in T configuration **(b) BSF in π configuration**

For a BSF (Band Stop Filter) we have the design values of L_1, C_1, L_2 and C_2 in terms of f_1, f_2 and R_0 as:

$L_1 = R_0 (f_2 - f_1) / \pi f_1 f_2$ henrys

$L_2 = R_0 / 4 \pi (f_2 - f_1)$ henrys

$C_1 = 1 / 4 \pi R_0 (f_2 - f_1)$ Farads

$C_2 = (f_2 - f_1) / \pi R_0 f_1 f_2$ Farads

Series arm inductance **L_1** $= R_0 (f_2 - f_1) / \pi f_1 f_2)$ henrys $= 600 (5000 - 2000) / (\pi \times 2000 \times 5000)$ H =

57.32 mH

and **$L_1/2 = 57.32 /2 = 28.66$ mH**

Shunt arm inductance **L_2** $= R_0 / 4 \pi (f_2 - f_1)$ henrys

$= 600 /4 \pi (5000 - 2000)$ H = **15.92 mH**

Next :

Series arm capacitance **C_1** $= 1 / 4 \pi R_0 (f_2 - f_1)$ Farads

$= 1 / 4 \pi \times 600 \times (5000 - 2000)$F = **0.044 μF**

and **$2C_1$** **=** 2×0.044 **= 0.088 μF**

Shunt arm capacitance **C_2** $= (f_2 - f_1) / \pi R_0 f_1 f_2$ Farads

$= (5000 - 2000) / (\pi \times 600 \times 2000 \times 5000)$ F = **0.159 μF**

Ex. 5 : Design a m derived Low Pass Filter in both T and π configurations having design Resistance of 600 Ω and to pass signals upto 1Khz with infinite attenuation frequency being 1.2 khz.

Solution: From the given data we have :

$F_{cLPF} = 1000$ Hz, $f_{inf} = 1200$ Hz and $R_0 = 600\ \Omega$

The basic configuration of a m derived LPF in both T and π configurations is shown below :

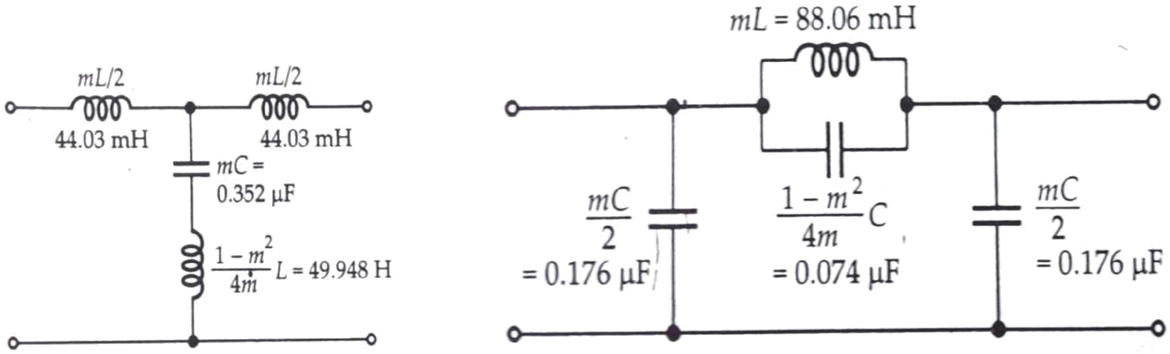


Fig: m derived LPF (a) In T configuration (b) In π configuration

From the theory of m derived LPF we have :

$$m = \sqrt{1-\frac{f_c^2}{f_\infty^2}} = \sqrt{1-(1000/1200)^2} = 0.553$$

For the proto type LPF the design values of L and C in terms of f_c and R_0 are given by :

$L = R_0 / \pi . f_c$ and
$C = 1/\pi . R_0 . f_c$

and using them we get :

Series arm inductance **L** = $R_0 / \pi f_c$ = 500/ π.1000 Henrys = **159.24 mH** and
Shunt arm capacitance **C** = $1/\pi R_0 . f_c$ = 1/ πx500x1000 Farads = **0.637 μF**

The values of the elements of T section of m derived LPF are :

mL/2 = (0.553 x 159.24) / 2 mH = **44.03 mH**

mC = 0.553 x 0.637 μF = **0.352 μF**

$[(1-m^2)/4m]$ L = $[(1-0.553^2)/4 \times 0.553]$ x 159.24 mH = **49.948 mH**

And the values of the elements of π section of m derived LPF are :

mC/2 = (0.553 x 0.637) /2 μF = **0.176 μF**

mL	= 0.553 x 159.24 mH	= **88.06 mH**
[(1-m^2)/4m] C	= [(1-0.553 ^{2})/4x0.553] x 0.637 µF	= **0.074 µF**

The required m derived LPF in T and π configurations is obtained by substituting these values in the basic configuration of m derived LPF in T and π configurations shown above.

Ex. 6 : Design a m derived High Pass Filter in both T and π configurations having design Resistance of 600 Ω and to pass signals beyond 4Khz with infinite attenuation frequency being 3.6 khz.

Solution: From the given data we have :

F_{cHPF} = 4000 Hz, f_{inf} = 3600 Hz and R_0 = 600 Ω

The basic configuration of a m derived High Pass Filter in both T and π configurations is shown in the figure below.

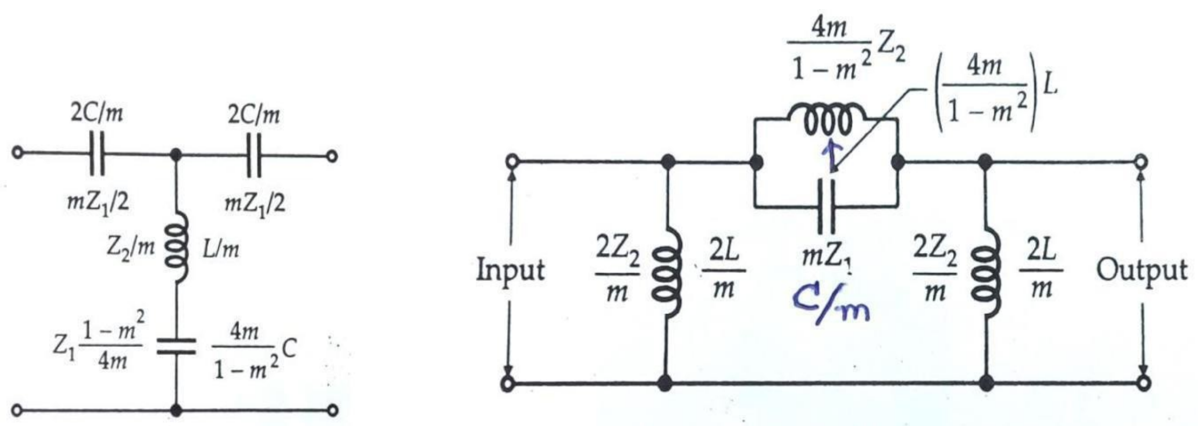


Fig: m derived High Pass Filter in (a) T configuration (b) π configuration

From the theory of m derived HPF we have :

= √1-(3600/4000)^2 = 0.436

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} = \sqrt{1-\left(\frac{\omega_\infty}{\omega_C}\right)^2}$$

For the proto type HPF the design values of L and C and R_0 are given in terms of f_c by :

$L = R_0 / 4 \pi . f_c$

and

$C = 1/4\pi . R_0 . f_c$

and using them we get :

Shunt arm inductance **L** = $R_0 / \pi f_c$ = 600/ 4π x 4000 Henrys = **11.94 mH** and

Series arm capacitance **C** = $1/4\pi R_0 . f_c$ = 1/ 4π x 600 x 4000 Farads = **0.033 μF**

The values of the elements of T section of m derived HPF are :

L/m	= 11.94/0.436 mH	= **27.39 mH**
2C/m	= (2 x 0.033) / 0.436 μF	= **0.151 μF**
$[4m /(1-m^2)]$ C	= $[4 \times 0.436 / (1-0.436^2)] \times 11.94$ **mH**	= **0.071 mH**

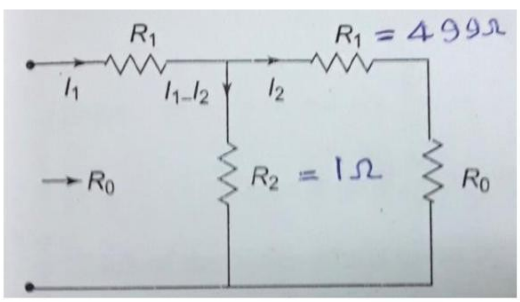
And the values of the elements of π section of m derived LPF are :

2L/m	= 2 x 11.94/0.436 mH	=**54.78 mH**
C/m	= 0.033 / 0.436 μF	=**0.076 μF**
$[4m /(1-m^2)]$ L	= $[4 \times 0.436 / (1-0.436^2)] \times 11.94$ **mH**	= **25.7 mH**

The required m derived HPF in T and π configurations is obtained by substituting these values in the basic configuration of m derived LPF in T and π configurations shown above.

Ex.7: Design a T type symmetrical attenuator with an attenuation of 60 db to work in a line of of 500 Ω Impedance.

The basic configuration of a T type attenuator is shown in the figure below:



The attenuation N of an attenuator is given by N = antilog (D / 20)

= antilog (60 / 20) = 1000

Each of the series arm Resistance R_1 of the symmetrical T attenuator is given by :

R_1 = R_0 . (N -1) / (N+1) = 500(1000 -1)/(1000 + 1) = **499 Ω**

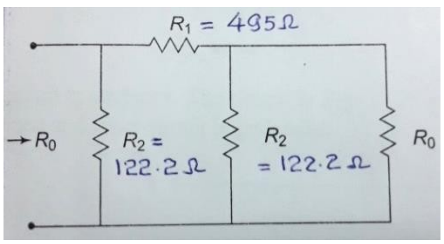
Shunt arm resistor R_2 is given by :

R_2 = R_0 . 2N / (N^2-1) = 500 x (2 x 1000) / (1000^2 - 1) = **1 Ω**

These values are substituted in the figure.

Ex.8: Design a π type symmetrical attenuator with an attenuation of 20 db to work in a line of 500 Ω Characteristic Impedance.

The basic configuration of a π type attenuator is shown in the figure below:



The attenuation N of an attenuator is given by N = antilog (D / 20)

= antilog (20 / 20) = 10

Series arm Resistance R_1 of the symmetrical π attenuator is given by :

R_1 = R_0 . (N^2-1) / 2N = 100(10^2 -1)/(2 x 10) = **495 Ω**

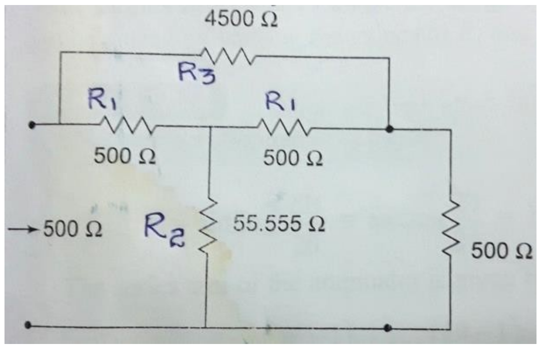
Each of the shunt arm resistor R_2 is given by :

$\mathbf{R_2} = R_0 \cdot (N+1) / (N-1) = 100 \times (10+1) / (10-1) = \mathbf{122.2\ \Omega}$

These values are substituted in the figure.

Ex.9: Design symmetrical bridged T attenuator with an attenuation of 20 db and terminated into a load of 500 Ω.

The basic configuration of a symmetrical bridged T attenuator is shown in the figure below:



From the given data we have D = 20 dB and R_0 = 500 Ω

The attenuation N of an attenuator is given by N = antilog (D / 20)

= antilog (20 / 20) = 10

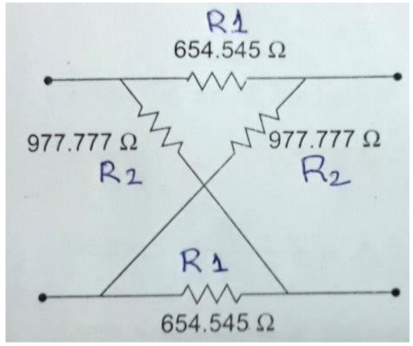
In a symmetrical bridged T attenuator the two series resistances (R_1) are always designed to be equal to R_0 and then the values of R_2 and R_3 are given by :

R_2	= R_0 / (N-1)	= 500 / (10-1)	= **55.5 Ω**
R_3	= R_0 (N-1)	= 500 (10-1)	= **4500 Ω**

These values are substituted in the figure.

Ex.10: Design a symmetrical lattice attenuator with an attenuation of 20 db and having a characteristic Impedance of 800 Ω.

The basic configuration of a symmetrical lattice attenuator is shown in the figure below:



From the given data we have D = 20 dB and R_0 = 500 Ω

The attenuation N of an attenuator is given by N = antilog (D / 20)

= antilog (20 / 20) = 10

The series and diagonal arm resistances R_1 and R_2 in a lattice attenuator are given by:

R_1	= R_0 (N-1) / (N+1)	= 800 (10-1) / 10+1)	= **654.5 Ω**
R_2	= R_0 (N+1) / (N-1)	= 800 (10+1) / (10-1)	= **977.7 Ω**

These values are substituted in the figure.

UNIT-4

LOCUS DIAGRAMS & RESONANCE

- **Locus Diagrams with variation of various parameters**
- **Series RC and RL circuits**
- **Parallel RLC circuits**
- **Resonance**
- **Series and Parallel circuits**
- **Concept of Bandwidth and Quality factor**

Locus Diagrams with variation of various parameters:

Introduction: In AC electrical circuits the magnitude and phase of the current vector depends upon the values of R,L&C when the applied voltage and frequency are kept constant. The path traced by the terminus (tip) of the current vector when the parameters R,L&C are varied is called the current ***Locus diagram*** . Locus diagrams are useful in studying and understanding the behavior of the RLC circuits when one of these parameters is varied keeping voltage and frequency constant.

In this unit,Locus diagrams are developed and explained for series RC,RL circuits and Parallel LC circuits along with their internal resistances when the parameters R,L and C are varied.

The term circle diagram identifies locus plots that are either circular or semicircular. The defining equations of such circle diagrams are also derived in this unit for series RC and RL diagrams.

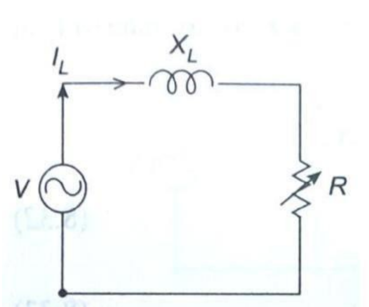
In both series RC,RL circuits and parallel LC circuits resistances are taken to be in series with L and C to highlight the fact that all practical L and C components will have at least a small value of internal resistance.

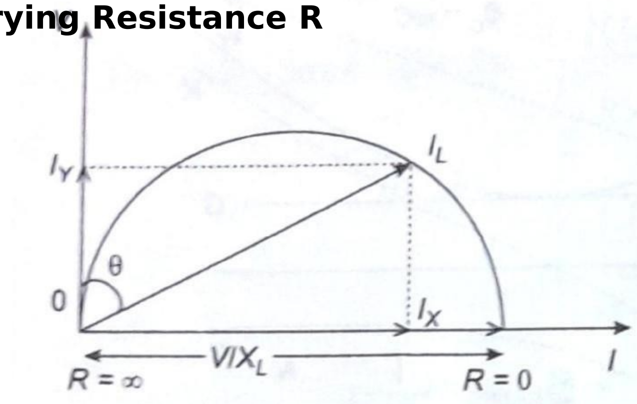
Series RL circuit with varying Resistance R:

Refer to the series RL circuit shown in the figure (a) below with constant X_L and varying R. The

current I_L lags behind the applied voltage V by a phase angle $\Theta = \tan^{-1}(X_L/R)$ for a given value of R as shown in the figure (b) below. When R=0 we can see that the current is maximum equal to V/X_L and lies along the I axis with phase angle equal to 90^0. When R is increased from zero to infinity the current gradually reduces from V/X_L to 0 and phase angle also reduces from 90^0 to 0^0. As can be seen from the figure, the tip of the current vector traces the path of a semicircle with its diameter along the +ve I axis.

Fig(a): Series RL circuit with Fig(b): Locus of current vector I_L with variation of R Varying Resistance R





Fig(a): Series RL circuit with Varying Resistance R

Fig(b): Locus of current vector I_L with variation of R

The related equations are:

$I_L = V/Z$ $\quad \sin\Theta = X_L/Z$ and $\cos\Theta = R/Z$ or $Z = X_L/\sin\Theta$

Therefore $I_L = (V/X_L)\sin\Theta$

For constant V and X_L the above expression for I_L is the polar equation of a circle with diameter (V/X_L) as shown in the figure above.

Circle equation for the RL circuit: (with fixed reactance and variable Resistance):

The X and Y coordinates of the current I_L are

$I_X = I_L \sin \Theta$ $I_Y = I_L \cos \Theta$

From the relations given above and earlier we get

$$I_X = (V/Z)(X_L/Z) = V X_L/Z^2 \quad \text{-------- (1)}$$

and

$$I_Y = (V/Z)(R/Z) = V R/Z^2 \quad \text{-------- (2)}$$

Squaring and adding the above two equations we get

$$I_X^2 + I_Y^2 = V^2(X_L^2+R^2) / Z^4 = (V^2Z^2)/Z^4 = V^2/Z^2 \quad \text{-------- (3)}$$

From equation (1) above we have $Z^2 = V X_L / I_X$ and substituting this in the above equation (3) we get :

$$I_X^2 + I_Y^2 = V^2 / (V X_L / I_X) = (V/X_L) I_X$$

Or

$$I_X^2 + I_Y^2 - (V/X_L) I_X = 0$$

Adding $(V/2X_L)^2$ to both sides ,the above equation can be written as

$$[I_X - V/2X_L]^2 + I_Y^2 = (V/2X_L)^2 \quad \text{----------- (4)}$$

Equation (4) above represents a circle with a radius of $(V/2X_L)$ and with it's coordinates of the centre as $(V/2X_L , 0)$

Series RC circuit with varying Resistance R:

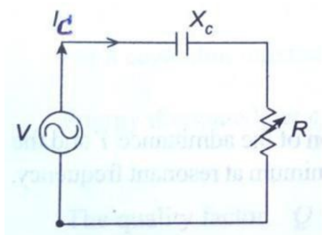
Refer to the series RC circuit shown in the figure (a) below with constant X_C and varying R. The current I_C leads the applied voltage V by a phase angle $\Theta = \tan^{-1}(X_C/R)$ for a given value of R as shown in the figure (b) below. When R=0 we can see that the current is maximum equal to V/X_C and lies along the negative I axis with phase angle equal to -90^0. When R is increased from zero to infinity the current gradually reduces from $-V/X_C$ to 0 and phase angle also reduces from -90^0 to 0^0. As can be seen from the figure, the tip of the current vector traces the path of a semicircle but now with its diameter along the negative I axis.

Circle equation for the RC circuit: (with fixed reactance and variable Resistance):

In the same way as we got for the Series RL circuit with varying resistance we can get the circle equation for an RC circuit with varying resistance as :

$$[I_X + V/2X_C]^2 + I_Y^2 = (V/2X_C)^2$$

whose coordinates of the centre are ($V/2X_C$, 0) and radius equal to $V/2X_C$ −



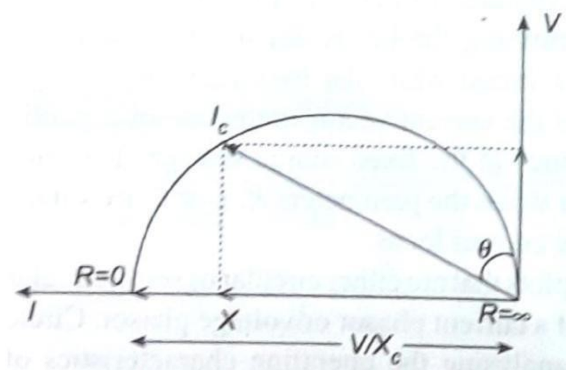


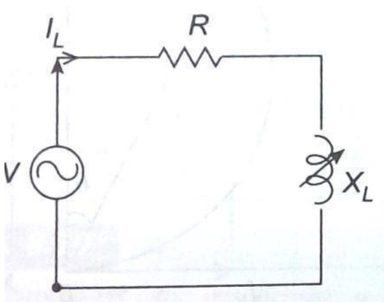
Fig: Series RC circuit with Varying Resistance R

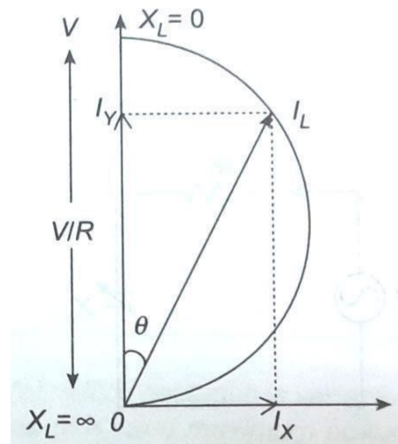
Fig: Locus of current vector I_C

with variation of R

Series RL circuit with varying Reactance X_L:

Refer to the series RL circuit shown in the figure (a) below with constant R and varying X_L. The current I_L lags behind the applied voltage V by a phase angle $\Theta = \tan^{-1}(X_L/R)$ for a given value of R as shown in the figure (b) below. When $X_L = 0$ we can see that the current is maximum equal to V/R and lies along the +ve V axis with phase angle equal to 0^0. When X_L is increased from zero to infinity the current gradually reduces from V/R to 0 and phase angle increases from 0^0 to 90^0. As can be seen from the figure, the tip of the current vector traces the path of a semicircle with its diameter along the +ve V axis and on to its right side.

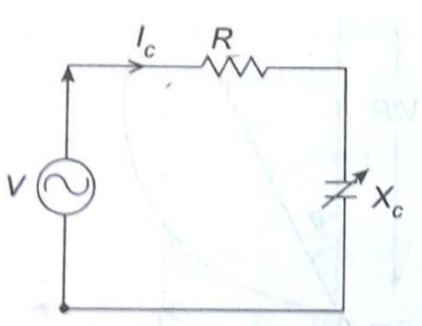


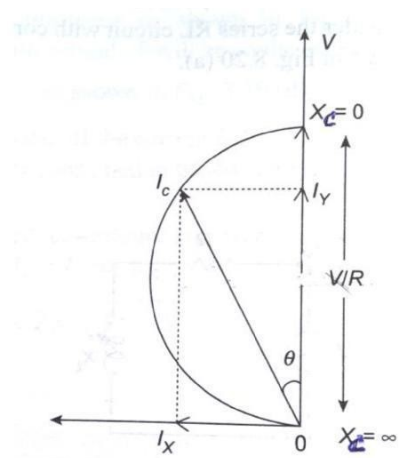


Fig(a): Series RL circuit with varying X_L **Fig(b) : Locus of current vector I_L with variation of X_L**

Series RC circuit with varying Reactance X_C:

Refer to the series RC circuit shown in the figure (a) below with constant R and varying X_C. The current I_C leads the applied voltage V by a phase angle $\Theta = \tan^{-1}(X_C/R)$ for a given value of R as shown in the figure (b) below. When $X_C = 0$ we can see that the current is maximum equal to V/R and lies along the V axis with phase angle equal to 0^0. When X_C is increased from zero to infinity the current gradually reduces from V/R to 0 and phase angle increases from 0^0 to 90^0. As can be seen from the figure, the tip of the current vector traces the path of a semicircle with its diameter along the +ve V axis but now on to its left side.



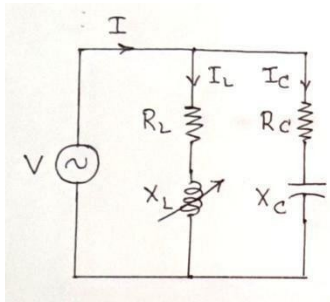


Fig(a): Series RC circuit with varying X_C **Fig(b): Locus of current vector I_C with variation of X_C**

Parallel LC circuits:

Parallel LC circuit along with its internal resistances as shown in the figures below is considered here for drawing the locus diagrams. As can be seen, there are two branch currents I_C and I_L along with the total current I. Locus diagrams of the current I_L or I_C (depending on which arm is varied)and the total current I are drawn by varying R_L, R_C , X_L and X_C one by one.

***Varying X_L*:**



Fig(a): parallel LC circuit with Internal Resistances R_L and R_C in series with L (Variable) and C (fixed) respectively.

The current I_C through the capacitor is constant since R_C and C are fixed and it leads the voltage vector OV by an angle $\Theta = \tan^{-1}$ (X /R) as shown in the figure (b). The current I through the inductance is the vector OI_L . It's amplitude is maximum and equal to V/R_L when X_L is zero and it is in phase with the applied voltage V. When X_L is increased from zero to infinity it's amplitude decreases to zero and phase will be lagging the voltage by 90^0 . In between, the phase angle will be lagging the voltage V by an angle $\Theta = \tan^{-1}$ (X /R). The locus of the current vector I is a semicircle with a diameter of length equal to V/R_L.

Note that this is the same locus what we got earlier for the series RL circuit with X_L varying except that here V is shown horizontally.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors staring point (we will take varying amplitude vector I_L)at the tip of the other vector (we will take constant amplitude vector I_C)and then join the start of fixed vector I_C to the end of varying vector I_L. Using this principle we can get the locus of the total current vector OI by shifting the I_L semicircle starting point O to the end of current vector OI_C keeping the two diameters parallel. The resulting semicircle I_CIB_T shown in the figure in dotted lines is the locus of the total current vector OI.

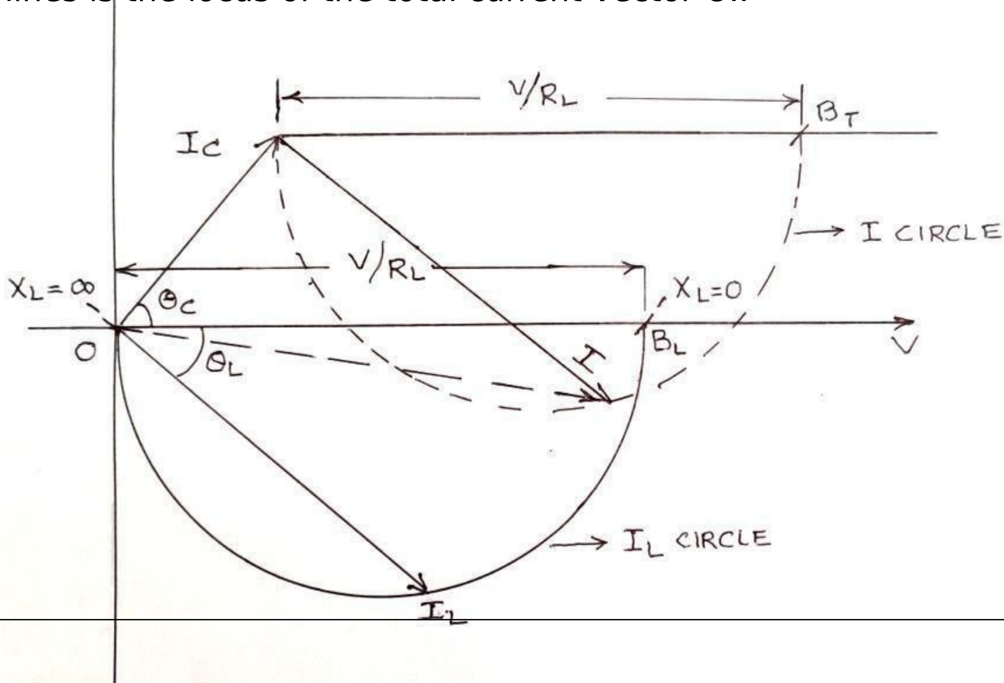


Fig (b): Locus of current vector I in Parallel LC circuit when X_L is varied from 0 to ∞

Varying X_C:

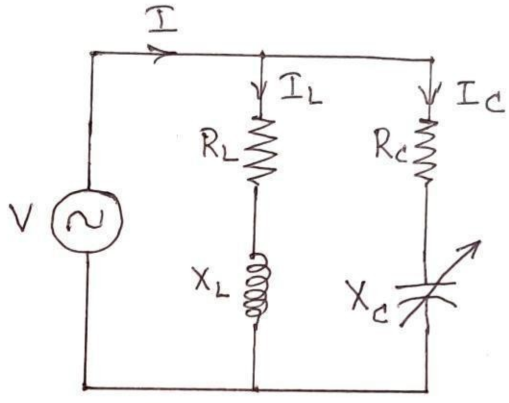
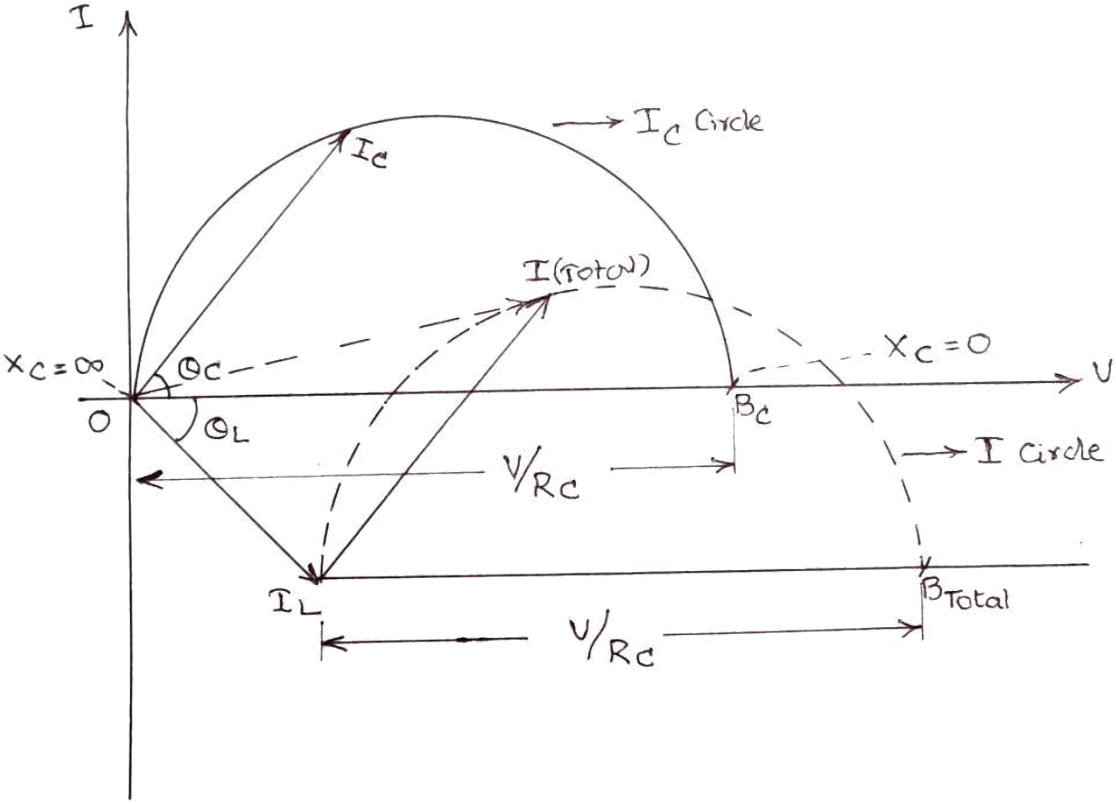


Fig.(a) parallel LC circuit with Internal Resistances R_L and R_C in series with L (fixed) and C (Variable) respectively.

The current I_L through the inductor is constant since R_L and L are fixed and it lags the voltage vector OV by an angle $\Theta = \tan^{-1}(X_L/R_L)$ as shown in the figure (b). The current I_C through the capacitance is the vector OI_C . It's amplitude is maximum and equal to V/R_C when X_C is zero and it is in phase with the applied voltage V. When X_C is increased from zero to infinity it's amplitude decreases to zero and phase will be leading the voltage by 90^0 . In between, the phase angle will be leading the voltage V by an angle $\Theta = \tan^{-1}(X_C/R_C)$. The locus of the current vector I_C is a semicircle with a diameter of length equal to V/R_C as shown in the figure below. Note that this is the same locus what we got earlier for the series RC circuit with X_C varying except that here V is shown horizontally.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors staring point (we will take varying amplitude vector I_C)at the tip of the other vector (we will take constant amplitude vector I_L) and then join the start of the fixed vector I_L to the end of varying vector I_C. Using this principle we can get the locus of the total current vector OI by shifting the I_C semicircle starting point O to the end of current vector OI_L keeping the two diameters parallel. The resulting semicircle $I_L I B_T$ shown in the figure in dotted lines is the locus of the total current vector OI.



Fig(b) : Locus of current vector I in Parallel LC circuit when XC is varied from 0 to ∞

***Varying RL*:**

The current IC through the capacitor is constant since RC and C are fixed and it leads the voltage vector OV by an angle Θ = $\tan^{-1}$ (X /R) as shown in the figure (b). The current I through the inductance is the vector OIL . It's amplitude is maximum and equal to V/XL when RL is zero. Its phase will be lagging the voltage by 90^{0}. When RL is increased from zero to infinity it's amplitude decreases to zero and it is in phase with the applied voltage V. In between, the phase angle will be lagging the voltage V by an angle Θ = $\tan^{-1}$ (X /R). The locus of the current vector IL is a semicircle with a diameter of length equal to V/RL. Note that this is the same locus what we got earlier for the series RL circuit with R varying except that here V is shown horizontally.

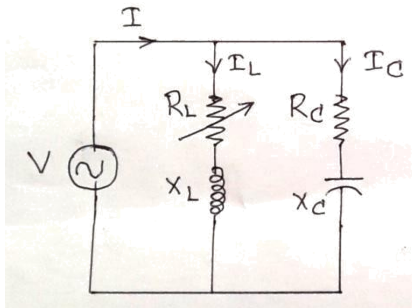
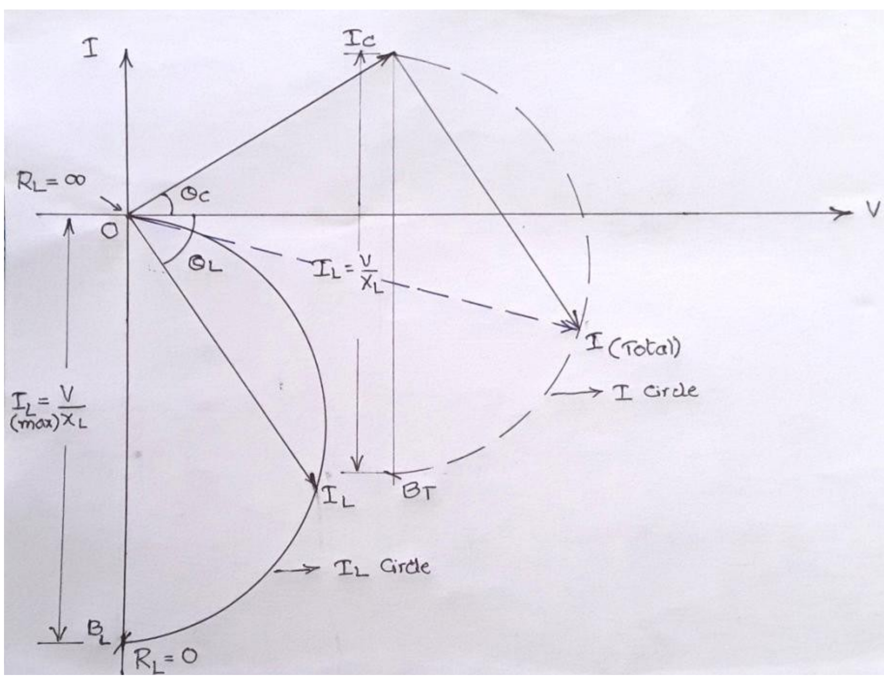


Fig.(a) parallel LC circuit with Internal Resistances R_L (Variable) and R_C (fixed) in series with L and C respectively.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors staring point (we will take varying amplitude vector I_L)at the tip of the other vector (we will take constant amplitude vector I_C)and then join the start of fixed vector I_C to the end of varying vector I_L. Using this principle we can get the locus of the total current vector OI by shifting the I_L semicircle starting point O to the end of current vector OI_C keeping the two diameters parallel. The resulting semicircle $I_C I B_T$ shown in the figure in dotted lines is the locus of the total current vector OI.



Fig(b) : Locus of current vector I in Parallel LC circuit when R_L is varied from 0 to ∞

***Varying R_C*:**

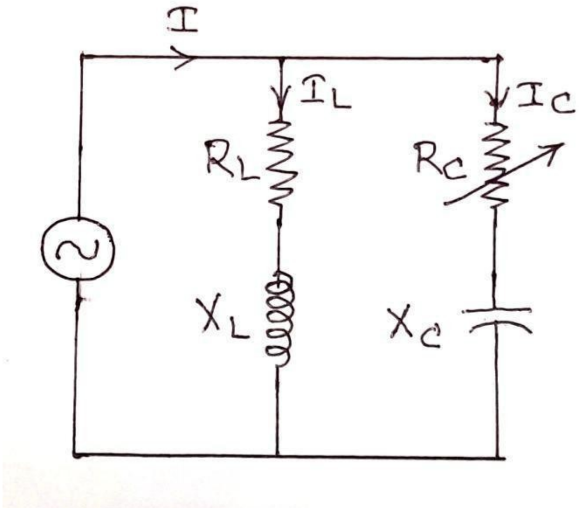
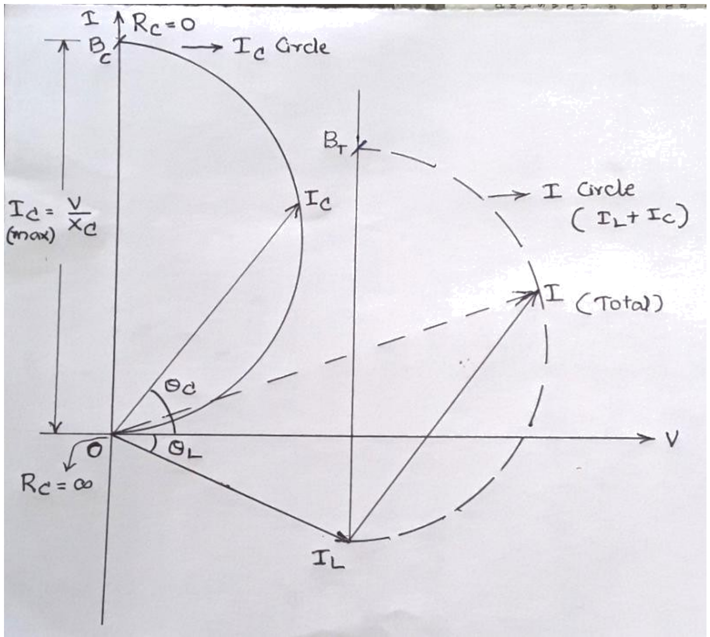


Fig.(a) parallel LC circuit with Internal Resistances R_L (fixed) and R_C (Variable) in series with L and C respectively.

The current I_L through the inductor is constant since R_L and L are fixed and it lags the voltage vector OV by an angle $\Theta = \tan^{-1}(X_L/R_L)$ as shown in the figure (b). The current I_C through the capacitance is the vector OI_C . It's amplitude is maximum and equal to V/X_C when R_C is zero and its phase will be leading the voltage by 90^0 . When R_C is increased from zero to infinity it's amplitude decreases to zero and it will be in phase with the applied voltage V. In between, the phase angle will be leading the voltage V by an angle $\Theta = \tan^{-1}(X_C/R_C)$. The locus of the current vector I_C is a semicircle with a diameter of length equal to V/X_C as shown in the figure below. Note that this is the same locus what we got earlier for the series RC circuit with R varying except that here V is shown horizontally.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors staring point (we will take varying amplitude vector I_C)at the tip of the other vector (we will take constant amplitude vector I_L) and then join the start of the fixed vector I_L to the end of varying vector I_C. Using this principle we can get the locus of the total current vector OI by shifting the I_C semicircle starting point O to the end of current vector OI_L keeping the two diameters parallel. The resulting semicircle I_LIB_T shown in the figure in dotted lines is the locus of the total current vector OI.



Fig(b) : Locus of current vector I in Parallel LC circuit when R_C is varied from 0 to ∞

Resonance :

Series RLC circuit:

The impedance of the series RLC circuit shown in the figure below and the current I through the circuit are given by :

$$\mathbf{Z = R + j\omega L + 1/j\omega C = R + j(\omega L - 1/\omega C)}$$

$$\mathbf{I = V_s/Z}$$

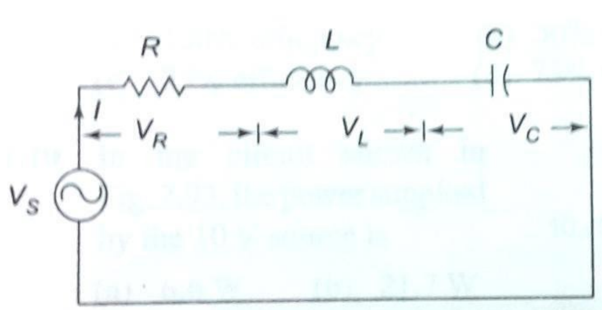


Fig: Series RLC circuit

The circuit is said to be in resonance when the Inductive reactance is equal to the Capacitive reactance. i.e. $\mathbf{X_L = X_C}$ **or** $\mathbf{\omega L = 1/\omega C}$**.** (i.e. Imaginary of the impedance is zero) The frequency

at which the resonance occurs is called resonant frequency. In the resonant condition when X_L

$= X_C$ they cancel with each other since they are in phase opposition(180^0 out of phase) and net impedance of the circuit is purely resistive.In this condition the magnitudes of voltages across the Capacitance and the Inductance are also equal to each other but again since they are of opposite polarity they cancel with each other and the entire applied voltage appears across the Resistance alone**.**

Solving for the resonant frequency from the above condition of Resonance **: ωL = 1/ωC** $2\pi f_r L = 1/2\pi f_r C$

$$f_r^2 = 1/4\pi^2 LC \text{ and } f_r = 1/2\pi\sqrt{LC}$$

In a series RLC circuit, resonance may be produced by varying L or C at a fixed frequency or by varying frequency at fixed L and C.

Reactances, Impedance and Resistance of a Series RLC circuit as a function of frequency: From the expressions for the Inductive and capacitive reactances we can see that when the frequency is zero, capacitance acts as an open circuit and Inductance as a short circuit. Similarly when the frequency is infinity inductance acts as an open circuit and the capacitance acts as a short circuit. The variation of Inductive and capacitive reactances along with Resistance R and the Total Impedance are shown plotted in the figure below.

As can be seen, when the frequency increases from zero to ∞ Inductive reactance X_L (directly proportional to ω) increases from zero to ∞ and Capacitive reactance X_C (inversely proportional to ω) decreases from _∞ to zero. Whereas, the Impedance decreases from ∞ to Pure Resistance R as the frequency increases from zero to f_r (as capacitive reactance reduces from _∞ and becomes equal to Inductive reactance) and then increases from R to ∞ as the frequency increases from f_r to ∞ (as inductive reactance increases from its value at resonant frequency to ∞)

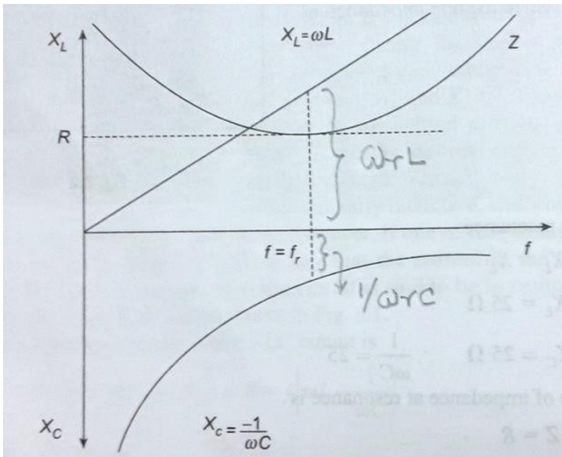


Fig : Reactance and Impedance plots of a Series RLC circuit

Phase angle of a Series RLC circuit as a function of frequency:

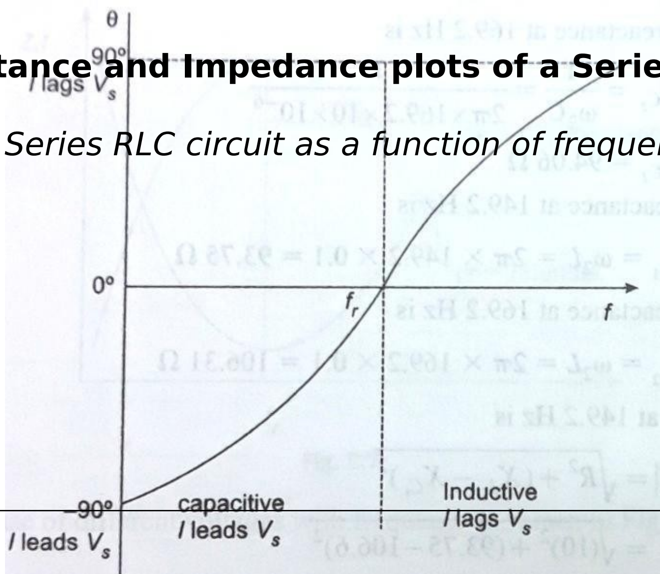


Fig : Phase plot of a Series RLC circuit

The following points can be seen from the Phase angle plot shown in the figure above:

- At frequencies below the resonant frequency capacitive reactance is higher than the inductive reactance and hence the phase angle of the current leads the voltage.
- As frequency increases from zero to **fr** the phase angle changes from -90^0 to zero. At frequencies above the resonant frequency
- inductive reactance is higher than the capacitive reactance and hence the phase angle of the current lags the voltage.
- As frequency increases from **fr** and approaches ∞ the phase angle increases from zero and approaches 90^0

Band width of a Series RLC circuit:

The band width of a circuit is defined as the Range of frequencies between which the output power is half of or 3 db less than the output power at the resonant frequency.These frequencies are called the cutoff frequencies, 3db points or half power points. But when we consider the output voltage or current, the range of frequencies between which the output voltage or current falls to 0.707 times of the value at the resonant frequency is called the Bandwidth BW. This is because voltage/current are related to power by a factor of √2 and when we are consider √2 times less it becomes 0.707. But still these frequencies are called as cutoff frequencies, 3db points or half power points. The lower end frequency is called ***lower cutoff frequency*** and the higher end frequency is called upper ***cutoff frequency.***

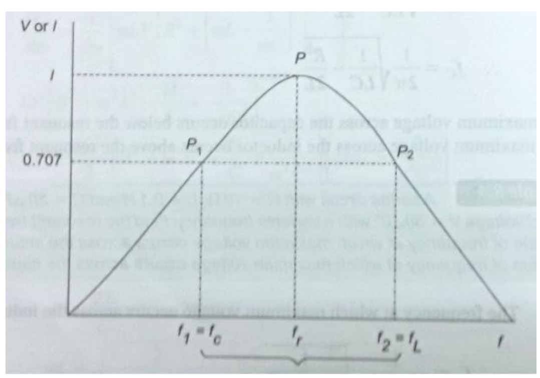


Fig: Plot showing the cutoff frequencies and Bandwidth of a series RLC circuit

Derivation of an expression for the BW of a series RLC circuit:

We know that BW = $f_2 – f_1$ Hz

If the current at points P_1 and P_2 are 0.707 (1/ 2) times that of I_{max} (current at the resonant frequency) then the Impedance of the circuit at points P_1 and P_2 is 2 R (i.e. 2 times the impedance at f_r)

But Impedance at point P_1 is given by: Z $= R^2 + (1/\omega_1 C – \omega_1 L)^2$ and equating this to 2 R we get : $(1/\omega_1 C) - \omega_1 L = R$ ------ (1)

Similarly Impedance at point P_2 is given by: $Z = R^2 + (\omega_2 L - 1/\omega_2 C)^2$ and equating this to 2 R we get: $\omega_2 L - (1/\omega_2 C) = R$ ------ (2)

Equating the above equations (1) and (2) we get:

$$1/\omega_1 C - \omega_1 L = \omega_2 L - 1/\omega_2 C$$

Rearranging we get $L(\omega_1 + \omega_2) = 1/C\,[(\omega_1 + \omega_2)/\omega_1\omega_2]$ i.e $\omega_1\omega_2 = 1/LC$

But we already know that for a series RLC circuit the resonant frequency is given by $\omega_r^2 = 1/LC$

Therefore: $\omega_1\omega_2 = \omega_r^2$ ---- (3) and $1/C = \omega_r^2 L$ ------ (4)

Next adding the above equations (1) and (2) we get:

$$1/\omega_1 C - \omega_1 L + \omega_2 L - 1/\omega_2 C = 2R$$

$$(\omega_2 - \omega_1)L + (1/\omega_1 C - 1/\omega_2 C) = 2R$$

$$(\omega_2 - \omega_1)L + 1/C[(\omega_2 - \omega_1)/\omega_1\omega_2) = 2R \quad \text{------} \quad (5)$$

Using the values of $\omega_1\omega_2$ and $1/C$ from equations (3) and (4) above into equation (5) above

we get: $(\omega_2 - \omega_1)L + \omega_r^2 L\,[(\omega_2 - \omega_1)/\omega_r^2) = 2R$

i.e. $2L(\omega_2 - \omega_1) = 2R$ i.e. $(\omega_2 - \omega_1) = R/L$ and $\mathbf{(f_2 - f_1) = R/2\pi L}$ ----- (6)

Or finally Band width **BW = R/2πL** ----- (7)

Since f_r lies in the centre of the lower and upper cutoff frequencies f_1 and f_2 using the above equation (6) we can get:

$$\mathbf{f_1 = f_r - R/4\pi L} \text{------} \quad (8)$$

$$\mathbf{f_2 = f_r + R/4\pi L} \text{------} \quad (9)$$

Further by dividing the equation (6) above by $\mathbf{f_r}$ on both sides we get another **important** relation : $\mathbf{(f_2 - f_1) / f_r = R/2\pi f_r L}$ **or BW / fr = R/2π fr L** ------- (10)

Here an important property of a coil i.e. **Q factor** or **figure of merit** is defined as the ratio of the reactance to the resistance of a coil.

$$\mathbf{Q = 2\pi f_r L / R} \text{-------} \quad (11)$$

Now using the relation (11) we can rewrite the relation (10) as

$$\mathbf{Q = f_r / BW} \text{-------} \quad (12)$$

Quality factor of a series RLC circuit:

The quality factor of a series RLC circuit is defined as:

Q = Reactive power in Inductor (or Capacitor) at resonance / Average power at Resonance

Reactive power in Inductor at resonance = $I^2 X_L$

Reactive power in Capacitor at resonance = $I^2 X_C$

Average power at Resonance = $I^2 R$

Here the power is expressed in the form I^2X (not as V^2/X) since I is common through R,L and C in the series RLC circuit and it gets cancelled during the simplification.

Therefore $\mathbf{Q = I^2 X_L / I^2 R} = \mathbf{I^2 X_C / I^2 R}$

i.e. $\mathbf{Q = X_L / R = \omega_r L / R}$ ------ (1)

Or $\mathbf{Q = X_C / R = 1/\omega_r RC}$ ------ (2)

From these two relations we can also define Q factor as :

Q = Inductive (or Capacitive) reactance at resonance / Resistance

Substituting the value of $\omega_r = 1/\sqrt{LC}$ in the expressions (1) or (2) for **Q** above we can get the value of **Q** in terms of **R, L,C** as below.

$$Q = (1/\sqrt{LC})\, L/R \qquad = (1/R)\,(\sqrt{L/C})$$

Selectivity:

Selectivity of a series **RLC** circuit indicates how well the given circuit responds to a given resonant frequency and how well it rejects all other frequencies. i.e. the selectivity is directly proportional to **Q** factor. A circuit with a good selectivity (or a high **Q** factor) will have maximum gain at the resonant frequency and will have minimum gain at other frequencies .i.e. it will have very low band width. This is illustrated in the figure below.

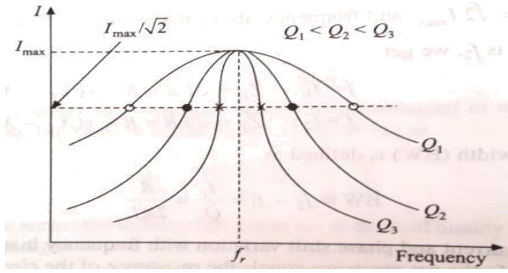


Fig: Effect of quality factor on bandwidthVoltage Magnification at resonance:

At resonance the voltages across the Inductance and capacitance are much larger than the applied voltage in a series RLC circuit and this is called voltage magnification at Resonance. The voltage magnification is equal to the **Q** factor of the circuit. This is proven below.

If we take the voltage applied to the circuit as **V** and the current through the circuit at resonance as **I** then

The voltage across the inductance **L** is: $V_L = IX_L = (V/R)\ \omega_r L$ and

The voltage across the capacitance **C** is: $V_C = IX_C = V/R\ \omega_r C$

But we know that the **Q** of a series RLC circuit = $\omega_r L / R = 1/R\ \omega_r C$ Using these relations in the expressions for V_L and V_C given above we get

$V_L = VQ$ and $V_C = VQ$

The ratio of voltage across the Inductor or capacitor at resonance to the applied voltage in a series RLC circuit is called Voltage magnification and is given by

$$\text{Magnification} = Q = V_L/V \quad \text{or } V_C / V$$

Important points In Series RLC circuit at resonant frequency :

- The impedance of the circuit becomes purely resistive and minimum i.e **Z = R** The current in the circuit becomes
- maximum
- The magnitudes of the capacitive Reactance and Inductive Reactance become equal The voltage across the Capacitor becomes equal to the
- voltage across the Inductor at resonance and is **Q** times higher than the voltage across the resistor

Bandwidth and Q factor of a Parallel RLC circuit:

Parallel RLC circuit is shown in the figure below. For finding out the **BW** and **Q** factor of a parallel RLC circuit, since it is easier we will work with Admittance , Conductance and Susceptance instead of Impedance ,Resistance and Reactance like in series RLC circuit.

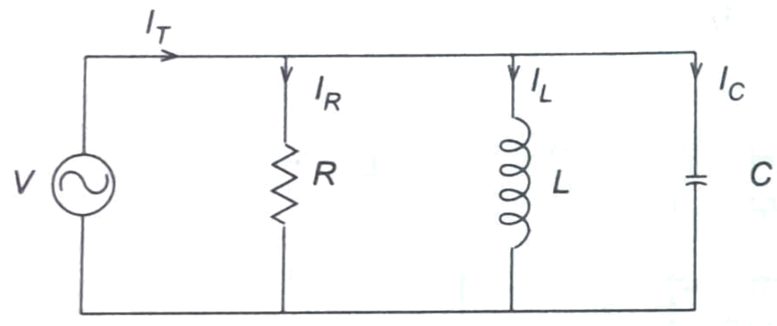


Fig: Parallel RLC circuit

Then we have the relation: $\mathbf{Y = 1/Z = 1/R + 1/j\omega L + j\omega C = 1/R + j\,(\omega C - 1/\omega L)}$

For the parallel RLC circuit also, at resonance, the imaginary part of the Admittance is zero and hence the frequency at which resonance occurs is given by: $\mathbf{\omega_r C - 1/\omega_r L = 0}$ **.** From this we get : $\mathbf{\omega_r C = 1/\omega_r L}$ **and** $\mathbf{\omega_r = 1/\sqrt{LC}}$ which is the same value for $\boldsymbol{\omega_r}$ as what we got for the series RLC circuit.

At resonance when the imaginary part of the admittance is zero the ***admittance*** becomes ***minimum***.(i.e ***Impedance*** becomes ***maximum*** as against Impedance becoming minimum in series RLC circuit) i.e. Current becomes minimum in the parallel RLC circuit at resonance (as against current becoming maximum in series RLC circuit) and increases on either side of the resonant frequency as shown in the figure below.

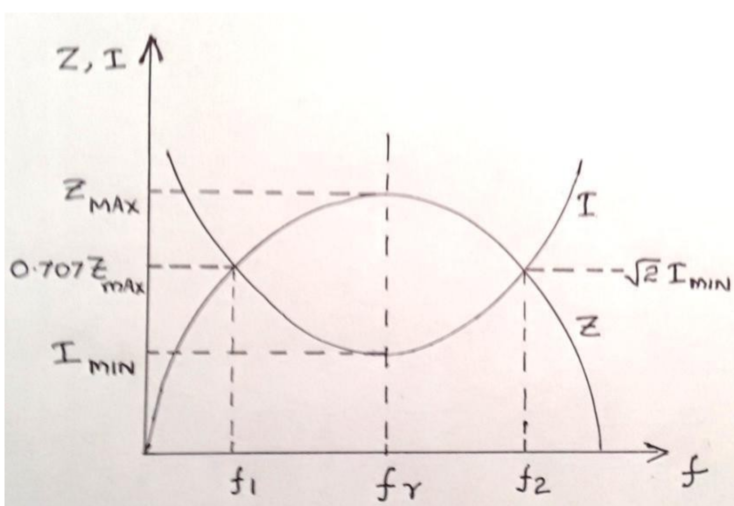


Fig: Variation of Impedance and Current with frequency in a Parallel RLC circuit

Here also the BW of the circuit is given by BW = f_2-f_1 where f_2 and f_1 are still called the upper and lower cut off frequencies but they are 3db higher cutoff frequencies since we notice that at these cutoff frequencies the amplitude of the current is $\sqrt{2}$ times higher than that of the amplitude of current at the resonant frequency.

The BW is computed here also on the same lines as we did for the series RLC circuit:

If the current at points P_1 and P_2 is $\sqrt{2}$ (3db) times higher than that of I_{min}(current at the

resonant frequency) then the admittance of the circuit at points P_1 and P_2 is also 2 times higher than the admittance at f_r)

But amplitude of admittance at point P_1 is given by: $Y = \sqrt{1/R^2 + (1/\omega_1 L - \omega_1 C)^2}$ and equating this to $\sqrt{2}/R$ we get

$$1/\omega_1 L - \omega_1 C = 1/R \quad \text{------} \quad (1)$$

Similarly amplitude of admittance at point P_2 is given by: $Y = \sqrt{1/R^2 + (\omega_2 C - 1/\omega_2 L)^2}$ and equating this to $\sqrt{2}/R$ we get

$$\omega_2 C - 1/\omega_2 L = 1/R \quad \text{------} \quad (2)$$

Equating LHS of (1) and (2) and further simplifying we get

$$1/\omega_1 L - \omega_1 C = \omega_2 C - 1/\omega_2 L$$

$$1/\omega_1 L + 1/\omega_2 L = \omega_1 C + \omega_2 C$$

$$1/L\,[(\omega_1 + \omega_2)/\,\omega_1\omega_2] = (\omega_1 + \omega_2)C$$

$$1/LC = \omega_1\omega_2$$

Next adding the equations (1) and (2) above and further simplifying we get

$$1/\omega_1 L - \omega_1 C + \omega_2 C - 1/\omega_2 L = 2/R$$

$$(\omega_2 C - \omega_1 C) + (1/\omega_1 L - 1/\omega_2 L) = 2/R$$

$$(\omega_2 - \omega_1)C + 1/L\,[(\omega_2 - \omega_1)/\,\omega_1\omega_2] = 2/R$$

Substituting the value of $\omega_1\omega_2 = 1/LC$

$$(\omega_2 - \omega_1)C + LC/L\,[(\omega_2 - \omega_1)] = 2/R$$

$$(\omega_2 - \omega_1)C + C\,[(\omega_2 - \omega_1)] = 2/R$$

$$2C\,[(\omega_2 - \omega_1)] = 2/R$$

Or $[(\omega_2 - \omega_1)] = 1/RC$

From which we get the band width **BW = f_2-f_1 = 1/2π RC**

Dividing both sides by f_r we get : **$(f_2 - f_1)/f_r = 1/2\pi f_r RC$** **------(1)**

Quality factor of a Parallel RLC circuit:

The quality factor of a Parallel RLC circuit is defined as:

Q = Reactive power in Inductor (or Capacitor) at resonance / Average power at Resonance Reactive power in Inductor at resonance = ***V^2/X_L***

Reactive power in Capacitor at resonance = V^2/X_C Average power at Resonance = V^2/R

Here the power is expressed in the form V^2/X (not as I^2X as in series circuit) since V is common across R,L and C in the parallel RLC circuit and it gets cancelled during the simplification. Therefore $Q = (V^2/X_L) / (V^2/R) = (V^2/X_C) / (V^2/R)$

i.e. $Q = R/X_L = R/\omega_r L$ ----- (1)

Or $Q = R/X_C = \omega_r RC$ -----(2)

From these two relations we can also define **Q** factor as :

Q = Resistance /Inductive (or Capacitive) reactance at resonance

Substituting the value of $\omega_r = 1/\sqrt{LC}$ in the expressions (1) or (2) for **Q** above we can get the value of **Q** in terms of R, L,C as below.

$$Q = (1/\sqrt{LC})\,RC = R\sqrt{C/L}$$

Further using the relation $Q = \omega_r RC$ (equation 2 above) in the earlier equation (1) we got in

BW viz. $(f_2 - f_1)/f_r = 1/2\pi f_r RC$ we get : $(f_2 - f_1)/f_r = 1/Q$ or $Q = f_r / (f_2 - f_1) = f_r / BW$

i.e. In Parallel RLC circuit also the Q factor is inversely proportional to the BW.

Admittance, Conductance and Susceptance curves for a Parallel RLC circuit as a function of frequency :

- The effect of varying the frequency on the Admittance, Conductance and Susceptance of a parallel circuit is shown in the figure below.
- Inductive susceptance **B_L** is given by **B_L = - 1/ωL.** It is inversely proportional to the frequency ω and is shown in the in the fourth quadrant since it is negative.
- Capacitive susceptance **B_C** is given by **B_C = ωC.** It is directly proportional to the frequency ω and is shown in the in the first quadrant as OP .It is positive and linear. Net susceptance **B = B_C - B_L**
- and is represented by the curve **JK**. As can be seen it is zero at the resonant frequency **f_r**
- The conductance **G = 1/R** and is constant
- The total admittance **Y** and the total current **I** are minimum at the resonant frequency as shown by the curve **VW**

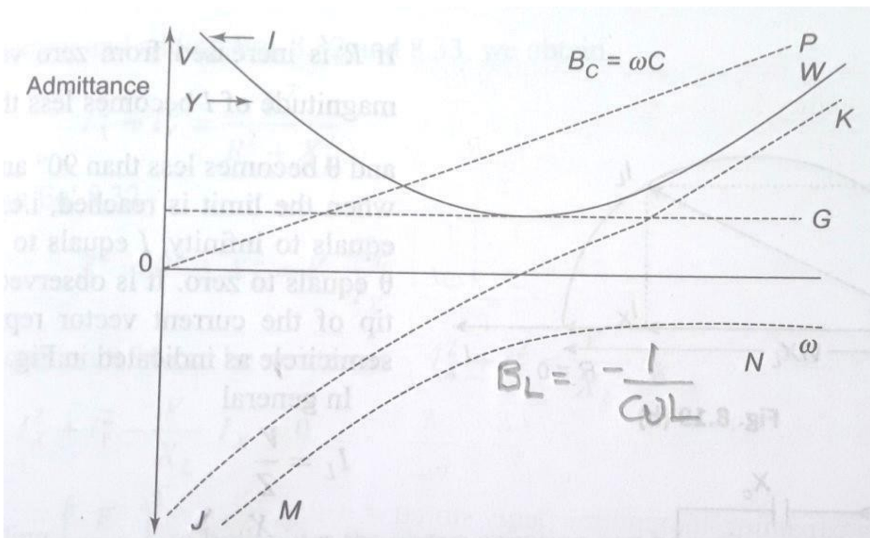


Fig: Conductance,Susceptance and Admittance plots of a Parallel RLC circuit

Current magnification in a Parallel RLC circuit:

Just as voltage magnification takes place across the capacitance and Inductance at the resonant frequency in a series RLC circuit , current magnification takes place in the currents through the capacitance and Inductance at the resonant frequency in a Parallel RLC circuit. This is shown below.

Voltage across the Resistance **= V = IR**

Current through the Inductance at resonance **I_L = V/ ω_r L = IR / ω_r L = I . R/ ω_r L = I Q**

Similarly

Current through the Capacitance at resonance **I_C = V/ (1/ω_r C) = IR / (1/ω_r C) = I(R ω_r C) = I Q**

From which we notice that the quality factor **$Q = I_L / I$ or I_C / I** and that the current through the inductance and the capacitance increases by **Q** times that of the current through the resistor at resonance. .

Important points In Parallel RLC circuit at resonant frequency :

- The impedance of the circuit becomes resistive and maximum i.e **Z = R**
- The current in the circuit becomes minimum
- The magnitudes of the capacitive Reactance and Inductive Reactance become equal
- The current through the Capacitor becomes equal and opposite to the current through the Inductor at resonance and is Q times higher than the current through the resistor

UNIT-5

DC MACHINES

- Principle of operation of DC Machines
- EMF Equation
- Types of Generators
- Magnetization and load characteristics of DC Generators
- DC Motors
- Types of DC motors
- Characteristics of DC Motors
- Speed control of DC shunt motor
- Flux and Armature control methods
- Losses and efficiency
- Swinburne’s test
    - Important concepts and Formulae
    - Illustrative Examples

Previous years question papers

Introduction:

A DC generator is a rotating machine which converts mechanical energy into DC electrical energy. It requires a prime mover like a Diesel engine, wind turbine or a steam turbine to rotate the DC generator. An EMF is induced in a DC Generator when there is a relative motion between a Magnetic field and a set of electrical conductors. The EMF induced is called a dynamically induced EMF or motional EMF . Normally the magnetic field is stationary and is obtained from stationary field coils placed on the Stator poles and the conductors are placed on a rotating shaft called Rotor. The basic constructional features of a DC generator and a DC Motor are same, and the same DC machine can work either as a DC generator or a DC motor.

The conversion of Mechanical energy into Electrical energy in DC generator is based on the principle of electromagnetic Induction. According to Faradays laws of Electromagnetic induction, whenever a conductor moves in a magnetic field a dynamically induced EMF is produced across the conductor. When the terminals of the conductor are connected to an electrical load the induced EMF enables a current flow through the load. Thus a mechanical energy in the form of a rotational motion given to a conductor is converted into Electrical energy. The EMF induced in a single conductor is very small. Hence a large set of conductors are used in practical generators and such a set of conductors placed on a rotating round shaft is called an armature.

Important parts and constructional features of a DC Generator:

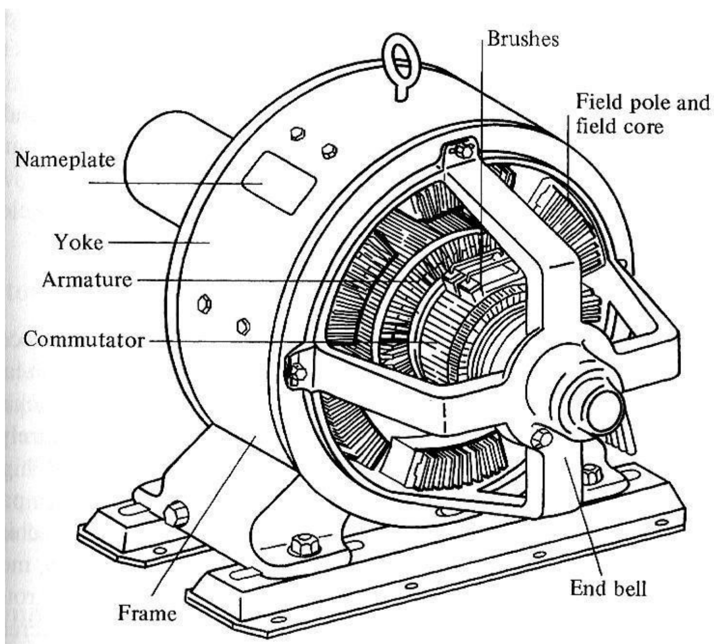


Fig: A simplified diagram of a dc machine:

Major parts of a DC generator:

- Main frame or Yoke
- Poles
- Armature
- Commutator
- Brushes ,bearings and shaft

The physical structure of the machine consists of two parts: the stator and the rotor.

The stationary part consists of the main frame (yoke), and the pole pieces, which project inward and provide a path for the magnetic flux. The ends of the pole pieces that are near the rotor spread out over the rotor surface to distribute its flux evenly over the rotor surface. These ends are called the pole shoes. The exposed surface of a pole shoe is called a pole face, and the distance between the pole face and the rotor is the air gap.

There are two principal windings on a dc machine:

- The armature windings: the windings in which a voltage is induced (rotor)
- The field windings: the windings that produce the main magnetic flux (stator)

Because the armature winding is located on the rotor, a dc machine's rotor is mostly called an armature. The terminal characteristic of a DC Machine is a plot of the output parameters of the Machine against each other. For a DC Generator the output quantities are the Terminal Voltage and the Line (Load) current.

Principle of operation of DC Machines:

Let us consider a single turn of coil **ABCD** mounted on a cylindrical shaft and rotated in an anticlockwise direction at constant angular velocity of **'ω'** rad/sec within a uniform magnetic field of flux density **B webers/mtrs^2** as shown in the figure below .

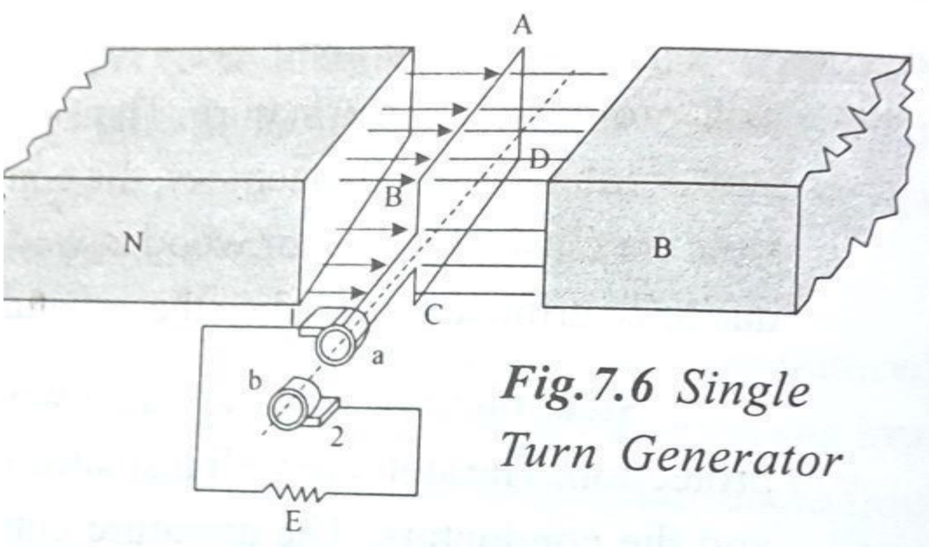


Fig.7.6 Single Turn Generator

Let **l** be the length and **b** be the breadth of the rectangular coil in meters. According to Faradays law the emf induced in a conductor is given by **e = N.dØ/dt** where **e** is the induced emf , **N** is the number of conductors **, Ø** is the flux linkage and **t** is the time. The flux linkage **Ø** is given by **: Ø = B.area of the coil.cos ωt = B.l.b.Cos ωt**

Since we are considering only one conductor the induced emf in the conductor is given by:

e = -dØ/dt = -d(B.l.b.Cos ωt)/dt = B.l.b.ω .Sin ωt = $\mathbf{E_m}$ **Sin ωt where** $\mathbf{E_m}$ **= B.l.b.ω**

As can be seen from the above equation for induced emf the voltage in a given generator can be increased by either increasing the flux density **'B'** or the rotational speed **'ω' .**

The induced emf '**e'**at any position of the coil as a function of time '**t'** as derived above is then given by : **e =** $\mathbf{E_m}$ **Sin ωt** where $\mathbf{E_m}$ **= B.l.b.ω**. As can be seen **dØ/dt** i.e rate of change of flux linkage is minimum (=0) when the coil is at perpendicular position to the flux lines and hence the induced voltage **e** is also minimum (=0) . We will call this as position $\boldsymbol{\Theta = 0^0}$ at the instant of say **t = 0 sec.** And **dØ/dt** is maximum when the coil is at parallel position to the flux lines and hence the induced voltage **e** is also maximum(**=** $\mathbf{E_m}$ **= B.l.b.ω**) and this position will then be $\boldsymbol{\Theta = 90^0}$ .When $\boldsymbol{\Theta = 180^0}$ the induced emf is again zero and when Θ $\mathbf{= 270^0}$ the emf induced is again maximum but now it would be negative. When **ϴ** = $\mathbf{360^0}$ the coil is back to the original position and the induced emf is again equal to zero. For the two pole generator shown in the figure one complete cycle of change takes place in one rotation of the coil. A plot of the induced emf 'e' **as** function of coil position **ϴ** is an alternating voltage as shown in the figure below.

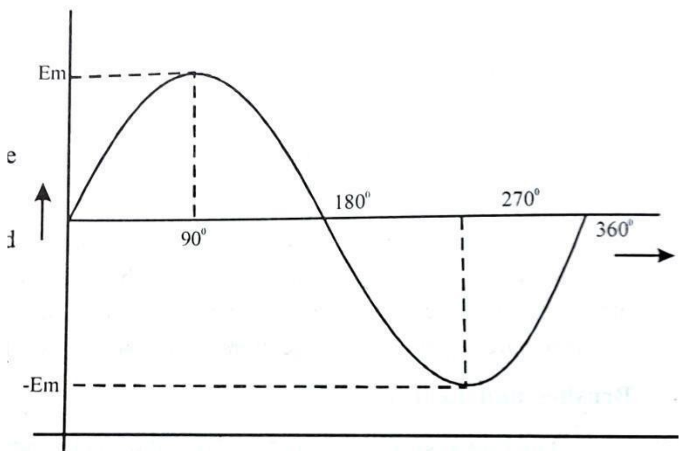


Fig: emf induced in a single turn generator in one full revolution

When the two terminals of the coil are connected to an external load (resistance in this case) through two separate rings (called slip rings) mounted on the armature current flows through the resistance and the current also would be sinusoidal.

The current flowing through the external load can be made unidirectional by replacing the two ***slip* rings** with two ***split* rings** as shown in the figure below.

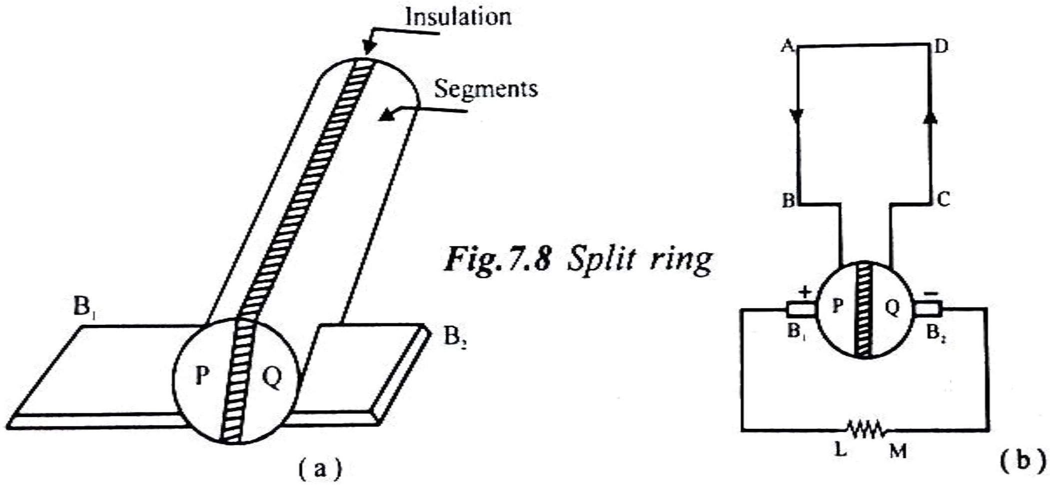


Fig.7.8 Split ring

One slip ring is split into two equal segments **P** and **Q** which are insulated from each other and the armature shaft. The two coils AB and CD are connected to the two segments **P** and **Q** .Two fixed (stationary) brushes **B1** and **B2** sliding along these two split rings will be collecting the current from the generator. During the first half of the revolution segment **P** is positive and current flows along **ABPLMQCD** through brush B1 which is positive and into brush B**2** into segment Q which is negative. Next during the other half cycle, the location of the segments **AB & CD** will reverse along with the respective segments **P and Q** . Now conductor **CD** and segment **Q** are positive and current flows along **DCQLMPBA** through the Brush **B1** which is again positive and into the brush **B2** which is again negative as shown in the figure below.

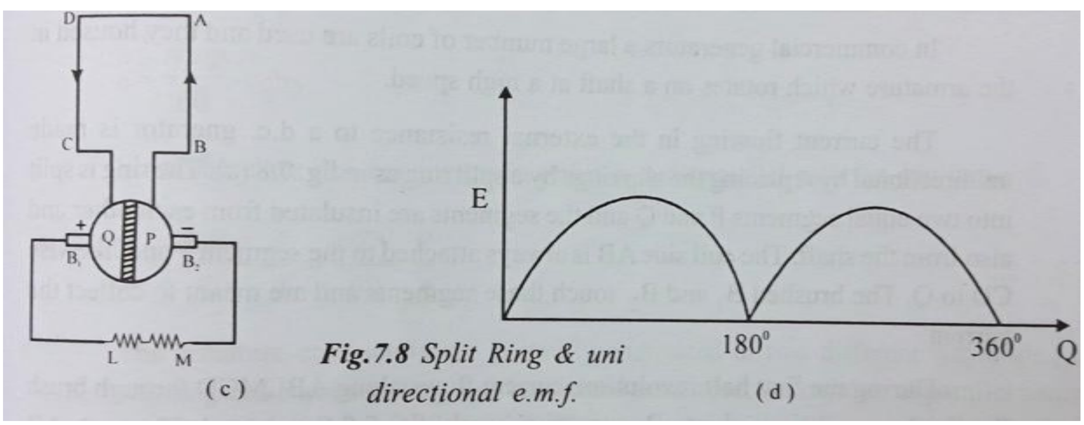


Fig.7.8 Split Ring & uni directional e.m.f.

In each half revolution the positions of the conductors **AB & CD** and the segments **P &Q** reverse but the brushes **B1&B2** are stationary and continue to collect current from the Positive side and deliver current to the Negative side respectively. Hence the voltage across the load will be a unipolar voltage as shown in the waveform above. The changeover of brushes **B1&B2** between segments **P &Q** takes place when the voltage is minimum so as to avoid or minimize the arcing between the split segments. In practical generators there will be more number of conductors and also more number of Pole pairs and hence more number of split segments are required and such a set of more number of split segments is called *commutator.*

EMF Equation:

Now we can derive a detailed expression for the exact induced emf in a generator in terms of all the following DC Machine parameters**.**

Ø The flux from a pole (webers)

Z The total number of conductors on the armature

a The number of parallel paths

- In a practical machine all the conductors are not connected in series. They are divided into groups of parallel conductors and then all the groups are connected in series to get higher voltage. In each group there are **'a'** conductors in parallel and hence there are **'a'** parallel current paths and each parallel path will have **Z/a** conductors in series.

N The Speed of rotation (RPM)

ω The speed (Radians/sec)

P The number of poles

Now consider one conductor on the armature. As this conductor makes one complete revolution it cuts **PØ** webers of flux.

Since the induced emf in a conductor is its rate of cutting of flux lines (Rate of change of Flux linkage) the emf **'e'** induced in such a single conductor is equal to

e = PØ/ Time for one revolution in seconds = PØ/(60/N) = NPØ/60 volts

There are **Z/a** conductors in series in each parallel path.

∴ the total induced emf **'E' = (Z/a) NPØ/60 = (NPØ Z)/ (a. 60)**

$$E_A = (Ø\ ZN/60).(P/a)$$

The armature conductors are generally connected in two methods. Viz. Lap winding and Wave winding**.**

In Lap wound machines the number of parallel paths **'a' = P** **'E' = (Ø ZN/60)**

In Wave wound machines the number of parallel paths **'a' = 2** **'E' = (Ø ZN/60).(P/2)**

In general the emf induced in a DC machine can be represented as **$E_A = K_a . Ø . N$**

where **$K_a = ZP/60.a$**

Sometimes it is convenient to express the emf induced in terms of the angular rotation **ω (Rad/sec)** and then the expression for emf becomes:

E_A = (Ø ZN/60).(P/a) = (ZP/a). Ø. N/60 = (ZP/.a). Ø. (ω/2π) =

(ZP/2πa).Ø.ω = K_a. Ø.ω (since N/60 RPS = 2π. N/60 Radians /sec

= ω Radians /sec and ∴ N/60 = ω/2π) Where **K_a** is the generalized

constant for the DC machine's armature and is given by :

$$K_a = (ZP/2\pi a)$$

Where **Ø** is the flux/per pole in the machine the angular speed (Radians/sec) and **K** parameters.

(Webers), **N** is the speed of rotation (RPM) **ω** is & **K_a** are constants depending on the machine

And thus finally ***E_A = Ka. Ø. ω*** *and we can say in general, the induced voltage in any DC machine depend on the following three factors:*

1. *The flux* ***Ø*** *in the machine*
2. *The angular speed of rotation* ***ω*** *and*
3. *A constant representing the construction of the machine.* ***(ZP/2πa)***

(i.e. the number of conductors ***'Z'****, the number of poles* ***'P'*** *and the number of parallel paths* ***'a'*** *along with the other constant* ***'2π'****)*

Important Aspects of DC Generators:

- The terminal characteristic of a DC Machine is a plot of the output quantities of the Machine against each other. For a DC Generator the output quantities are the Terminal Voltage and the Line (Load) current.
- The various types of Generators differ in their terminal characteristics (Voltage-Current) and therefore to the application to which they are suited.
- The DC Generators are compared by their Voltages, Power ratings, their efficiencies and Voltage

  regulation. Voltage Regulation **(VR)** is defined by the equation: **V_R = [(Vnl − Vfl) / Vfl].100 %**

  Where **Vnl** is the No load terminal voltage and **Vfl** is the Full load terminal voltage. It is a rough measure of the Generator's Voltage-Current Characteristic. A positive voltage regulation means a drooping characteristic and a negative regulation means a rising characteristic.
- Since the speed of the prime movers affects the Generator voltage and prime movers can have varying speed characteristics, The voltage regulation and speed characteristics of the Generator are always compared assuming that the Prime ***mover's speed is always constant***.

Magnetization characteristics of DC Generators:

No load or Open circuit magnetization characteristic of any DC Machine is a plot of the Field flux versus the magnetizing current. Since measurement of field flux is difficult we use the relation for the emf

induced in a DC machine **E_A = K. Ø.N** from which we can see that the induced voltage is proportional to the Flux in the machine when the speed is maintained constant. Hence we conduct a test on the given DC machine to obtain data on the induced voltage as a function of the field current.

The diagram of the test setup required to obtain the above data is shown in the figure below.

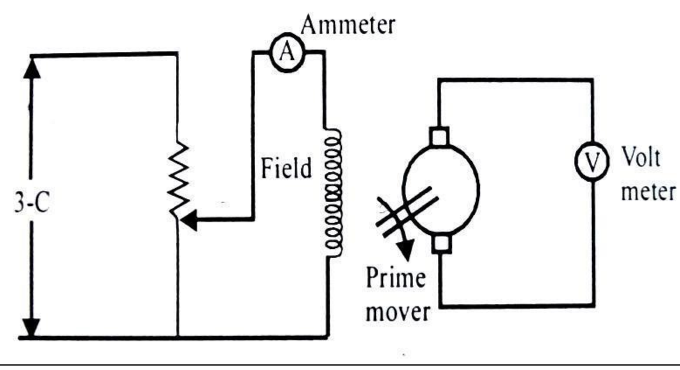


Fig: Test setup with a DC machine to obtain the No load magnetization Characteristic

The prime mover gives the required mechanical energy to the DC Machine and it can be a small Diesel engine. The rheostat connected between the DC Input and the field winding is used to adjust and get the required field current. The field current is initially set to Zero and the Armature volatage is measured. Then the field current is gradually increased and the corresponding values of Armature voltage are measured until the output voltage saturates. Next the field current is brought back to zero gradually and the corresponding Armature voltages are measured at a few points. The corresponding data on Armature voltage is plotted against field current and is shown in the figure below.

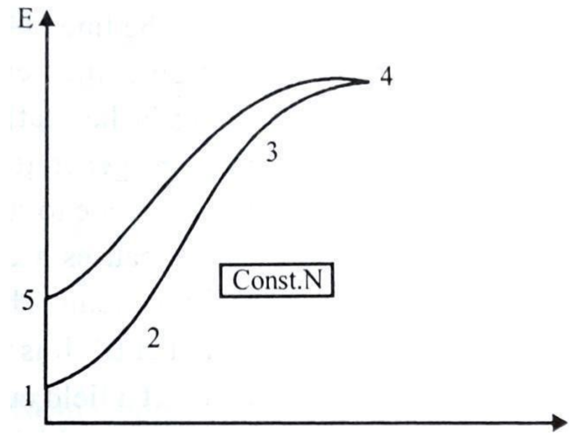


Fig: No load magnetization curve (or OCC) of a DC Machine (Plot of Armature Volatage Vs.field current)

Though the field current is zero we get a small value of Armature voltage as seen at point 1 due to the residual magnetism present in the field coil. Subsequently armature voltage increases with field current upto some point 3 and then the rate of rise decreses. Finally at poin 4 field flux gets saturated and hence the emf also gets saturated. The plot of armature voltage vs.field current is not same during the field current reduction as that during the field current increase and this is due to the property of magnetic hysteresis in the Ferro magnetic materials. In the return path the induced volatage at zero field current is higher than that during the field current increase. This is due to the combined effect of Hysterisis and the residual magnetism.

Different Types of DC Generators and their Terminal (or Load) Characteristics:

The DC generators are classified according to the manner in which the field flux is produced. Let us consider the following important types of DC Generators and their characteristics along with their equivalent circuits.

The following notation is used uniformly in all the following circuits/characteristics:

- V_T = Generator’s Terminal Voltage
- I_L = Load or line current
- I_A = Armature current
- E_A = Armature voltage
- R_A = Armature Resistance
- I_F = Field current
-

- V_F = Field voltage
- R_F = Field Resistance

Separately Excited Generator: In this type the field flux is derived from a separate power source which is independent of the Generator. The equivalent circuit of such a machine along with the governing equations is shown in the figure below.

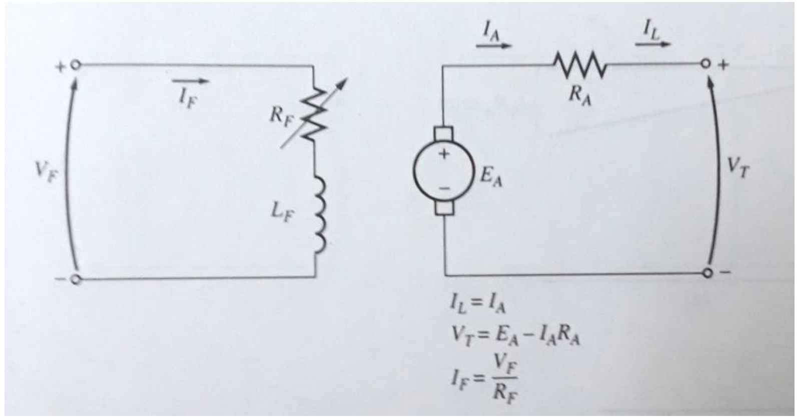


Fig: Equivalent circuit of a separately excited DC Generator

The terminal characteristic of this type of Generators is a plot of **V_T vs. I_L** for a constant speed **ω** and the governing equations are :

- The Load or line current **I_L** = The armature current **I_A**
- Generator's Terminal Voltage = **V_T** = (**E_A** – **$I_A R_A$**)
- **$I_F = V_F / R_F$**

Since the internally generated voltage is independent of **I_A**, the terminal characteristic of a Separately Excited Generator is a straight line as shown in the figure below.

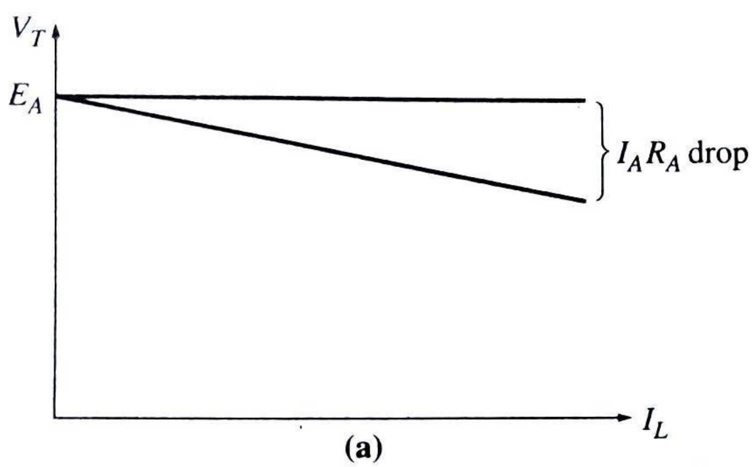
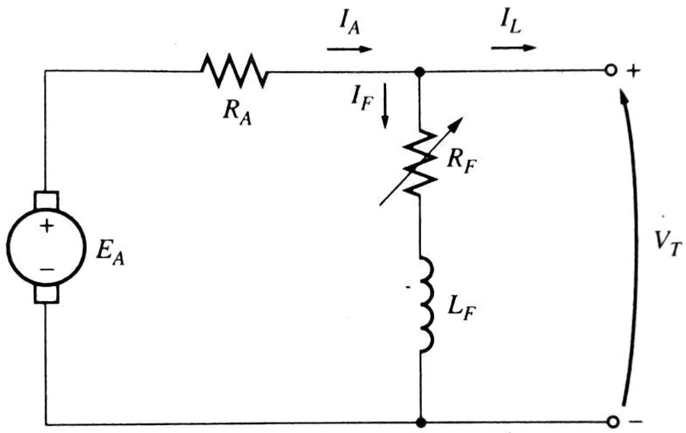


Fig: The terminal Characteristics of a Separately Excited DC Generator

When the load supplied by the generator increases, the load current **I_L** increases and hence the armature current **I_A** also increases. When the armature current increases, the **$I_A R_A$** drop increases, so the terminal voltage of the generator droops (falls). It is called a drooping characteristic.

Shunt Generator: In this the field flux is derived by connecting the Field directly across the Armature terminals. The equivalent circuit of such a generator is shown in the figure below along with the governing equations.



$$I_A = I_F + I_L$$

$$V_T = E_A - I_A R_A$$

$$I_F = \frac{V_T}{R_F}$$

Fig: The equivalent circuit of a DC Shunt generator along with the relevant governing equations

As could be seen, in this machine the armature current supplies both the load current and the field current. Using the Kirchhoff's voltage law the terminal voltage is seen to be same as that of a separately excited voltage i.e. $V_T = (E_A - I_A R_A)$. In this the advantage is that no external supply is required for the field circuit. *But this leaves an important question. If the generator supplies its own field current how does it get the initial field flux that is required to start the machine and generate voltage when it is first turned on? This is explained below.*

Voltage build up in a Shunt Generator:

The voltage build up in a shunt generator depends upon the presence of a ***residual flux*** in the poles of the generator. When a Shunt generator first starts to turn on an internal voltage is generated which is given by $E_A = k.\ Ø_{res}.\ \omega$. This voltage(which may be just one or two volts) appears at the generator terminals. This causes a current to flow in the generator's field coil $I_F = V_T / R_F$. This produces a m.m.f. in the poles which in turn increases the flux in them. The increase in the flux causes an increase in $E_A = k.Ø\uparrow.\ \omega$ which in turn increases the terminal voltage V_T . When V_T rises, I_F increases further, increasing the flux more which increases E_A and so on. This voltage build up phenomenon is shown in the figure below.

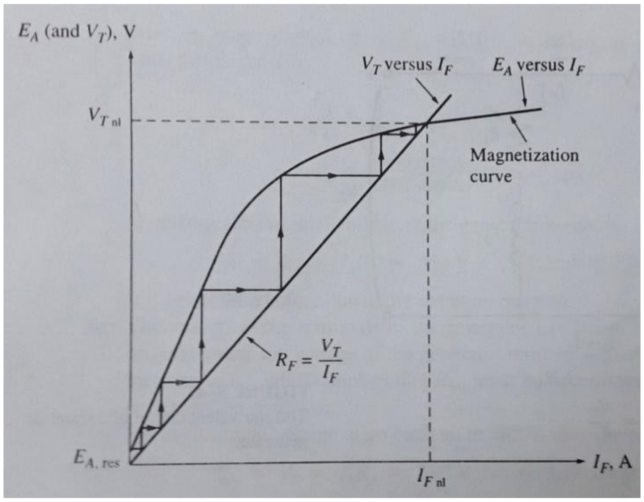


Fig: Voltage build up on starting in a DC Shunt generator

It is to be noted here that it is effect of ***magnetic saturation*** in the Pole faces which eventually limits the build of the terminal voltage.

The voltage build up in the figure above shows up as though it is building up in discrete steps. It is not so. These steps are shown just to make it clear the phenomenon of positive feedback between the Generator's internal voltage and the field current. In the DC Shunt generator both $\mathbf{E_A}$ and $\mathbf{I_F}$ increase simultaneously until the steady state conditions are reached.

The terminal characteristics of the shunt generator differ from that of the separately excited generator because the amount of field current depends on its terminal voltage. As the generator load is increased,

the load current $\mathbf{I_L}$ increases and so $\mathbf{I_A = I_F + I_L\uparrow}$ also increases . An increase in $\mathbf{I_A}$ increases the $\mathbf{I_AR_A}$ drop causing $\mathbf{V_T}$ = ($\mathbf{E_A -- I_A \uparrow R_A}$ **)** to decrease. This is precisely the same behavior we have seen in the case of separately excited generator.However,in the shunt generator when $\mathbf{V_T}$ decreases the field current decreases ,hence the field flux deceases thus decreasing the generated Voltage $\mathbf{E_A}$ **.** Decreasing the $\mathbf{E_A}$ causes a further decrease in the terminal voltage $\mathbf{V_T = (E_A\downarrow --I_A R_A)}$**.** The resulting characteristic is shown in the figure below.

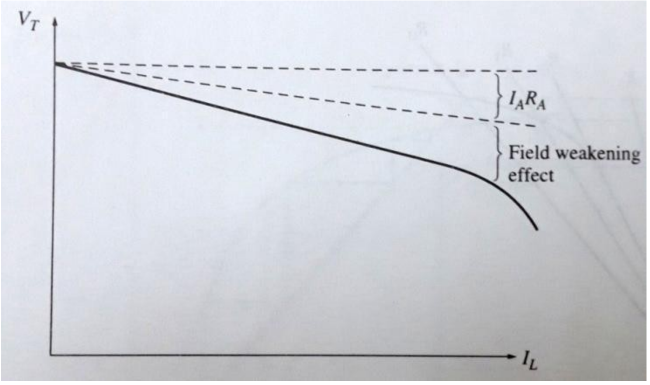


Fig: **Terminal Characteristic of DC Shunt Generator**

It can be noticed that the drop with load is steeper than that of a separately excited motor due to the field weakening affect. This means that the regulation of a Shunt Generator is worse than that of a Separately Excited Generator.

DC Series Generator: In this the field flux is derived by connecting the Field coil in series with the Armature of the Generator as shown in the figure below.

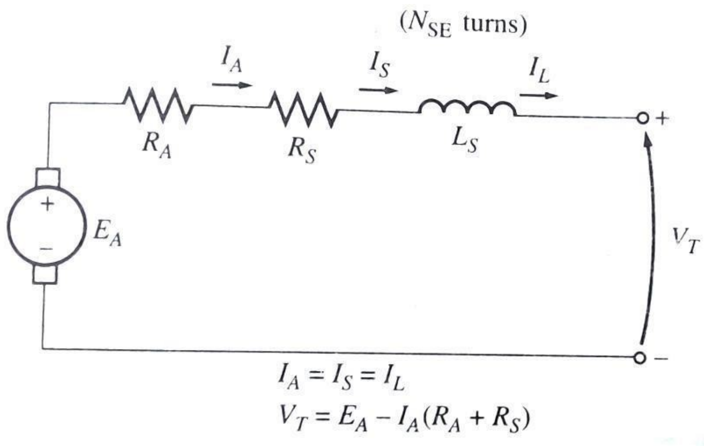


Fig: Equivalent circuit of DC Series Generator along with the governing equations

As shown the armature current, load current and field current are same in a DC series generator. i.e

$\mathbf{I_A = I_F = I_L}$. Since the mmf produced by the fields is given by **= NI** and the field current is more in the DC series generator , the fileld winding is wound with lesser number of turns and also with a thicker gauge so as to offer less field resistance since full load current flows through the field winding .

The terminal characteristic of a DC Series Generator looks very much like the magnetization curve of any other type of generator and is shown in the figure below.

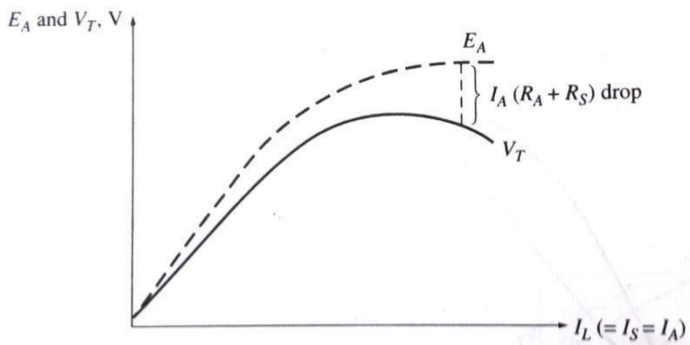


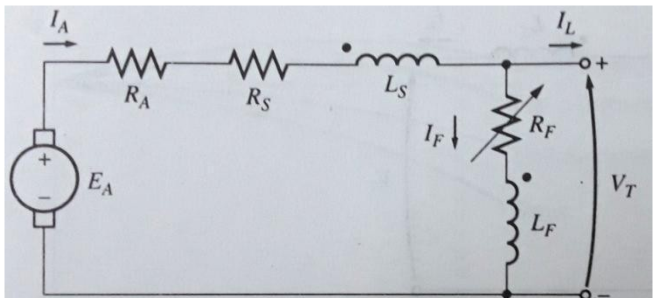
Fig: Terminal Characteristic of DC Series Generator

At no load however since there is no field current armature voltage $\mathbf{E_A}$ and also the terminal voltage $\mathbf{V_T}$ are very small (generated by the small amount of residual flux.) As the load increases ,field current rises hence $\mathbf{E_A}$ also increases rapidly. The

$\mathbf{I_A (R_A+R_F)}$ drop also goes up but this rise is less predominant compared to the rise in $\mathbf{E_A}$ initaially and hence $\mathbf{V_T}$ also rises initially. After some time field flux gets saturated and hence the induced voltage **E_A** will be constant without any further rise. At this stage the resistive drop predominates and hence the ***terminal voltage V_T starts drooping*****.**

DC Compound generator:

As we know in DC shunt Generator the terminal Voltage falls and in a DC series generator the terminal voltage increases on loading. A compound DC Generator is the one in which there will be both Series and shunt field coils. If they are wound such that they aid each other then it is called a Cumulative Compound DC Generator and if they are wound such that the two fields oppose each other, then it is called a differential Compound DC Generator. The equivalent circuit diagram of such Cumulative DC Generator along with relevant governing equations is shown in the figure below.



$$I_A = I_L + I_F$$
$$V_T = E_A - I_A(R_A + R_S)$$
$$I_F = \frac{V_T}{R_F}$$
$$\mathscr{F}_{\text{net}} = N_F I_F + N_{SE} I_A - \mathscr{F}_{AR}$$

Fig: Equivalent circuit of a Cumulative compound DC Generator

The circuit diagram is shown with standard ***dot convention*** on the field windings. ***i.e. The current flowing into the dot side of the winding produces a positive mmf .***

And as can be seen that both **I_F** in the shunt winding and **I_A** in the series winding flow into the dot side and hence both produce magnetic fields which are positive and hence aid each other.

When the two fields are aiding each other we get a characteristic which will have the combined effect of ***drooping*** (due to the shunt coil) and ***rising*** (due to the field coil). Whichever coil current is more its effect will be more predominant. The terminal characteristics of a cumulative compound DC Generator are shown in the figure below for all the three cases.

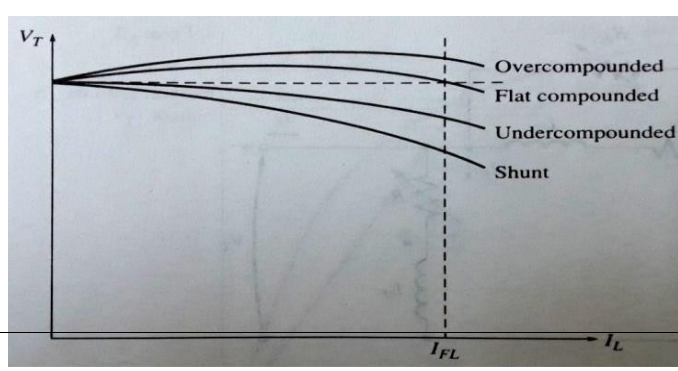


Fig: Terminal Characteristics of a DC Compound Generator

1. If the Series field effect is more dominating than that of the Shunt field coil then we get the ***Over compounded*** characteristic where the full load terminal voltage is higher than the no load terminal voltage.
2. If the Series field effect is equal to that of the Shunt field coil then we get the ***Flat compounded*** characteristic where the full load terminal voltage is equal to the no load terminal voltage.
3. If the Shunt field effect is more dominating than that of the Series field coil then we get the ***Under compounded*** characteristic where the full load terminal voltage is lower than the no load terminal voltage.

The normal shunt characteristic is also shown in the figure for comparison.

DC Motors:

***Principle of operation*:** DC Motors are DC machines used as motors. A DC Motor converts the input DC power into output rotational mechanical power from the following principle. A current carrying conductor placed in a magnetic field experiences a mechanical force given by **F= i (l X B).**

When a group of such conductors is placed on a rotor and are connected properly the force experienced by the all the conductors together gets translated into a torque on the rotor (armature) and it starts rotating. We will derive an expression for such a Torque developed by a DC Motor from the first principles and its equivalent circuit by equating the Electrical power given to the motor (excluding the losses) to the mechanical power developed by the motor. .

Torque developed by a DC Motor: The equivalent circuit of a DC motor is shown in the figure below.

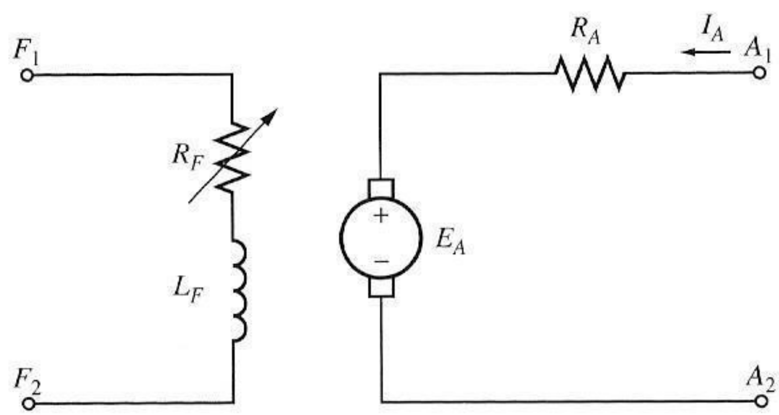


Fig: Equivalent circuit of a DC motor

In this figure, the armature circuit is represented by an ideal voltage source **E_A** and the armature resistance **R_A**. The field coils, which produce the magnetic flux in the motor, are represented by inductor **L_F** and the field resistance **R_F**. The separate external variable resistor used to control the amount of current in the field circuit is also combined with the field resistance and is together shown as **R_F**.

We know from the earlier study of generators that the voltage generated in a DC Machine when It is rotating in a magnetic flux of **Ø** webers/pole is given by **$E_A = K_A . Ø . \omega$** where **K_A** is given by:

$$K_A = (ZP/2\pi a)$$

Now in the DC Motor also, when it is rotating, from the same fundamental principle of Generator a Voltage is generated across the armature and it is now called back EMF and is normally shown as $\mathbf{E_b}$ to distinguish it from the voltage generated in the armature of a generator which was shown as $\mathbf{E_A}$ **.**

The governing equation of the DC Motor armature circuit now becomes:

$$\mathbf{V_T = E_b + I_a R_A \text{ or } E_b = V_T - I_a R_A}$$

(as against $\mathbf{V_T = E_A - I_a R_A}$ in the case of a generator where $\mathbf{I_A}$ flows from armature towards the external terminals i.e external load) since now an external voltage V_T is applied to the motor terminals , direction of armature current changes i.e. now it flows from external terminals towards the armature.

The power delivered to the motor is given by : $P_{in} = V_T \cdot I_a$. From this, the loss of power in the armature is equal to $I_a^2 R_A$ and hence the net power given to the motor armature is given by :

$$P_m = V_T \cdot I_a - I_a^2 R_A = I_a (V_T - I_a R_A) = I_a \cdot E_b$$

$$P_m = I_a \cdot E_b$$

This net electrical power is converted into mechanical power. We know that in mechanical rotational systems the power is equal to Torque times the speed. In the SI system of units which is the present Industry standard it is given by :

$$P_{mech} \text{ (watts)} = \tau \text{ (Nw.mtrs)}.\omega \text{ (Radians/second)}$$

For simplification if we ignore the mechanical losses in the motor,then :

$$P_m = I_a \cdot E_b = P_{mech} = \tau \cdot \omega$$

$$\text{i.e. } \tau \cdot \omega = I_a \cdot E_b = E_b \cdot I_a$$

Substituting the value of $E_A = K_A \cdot Ø \cdot \omega$ induced emfs we get $\tau \cdot \omega = I_a \cdot K_A \cdot Ø \cdot \omega$ we got in generators here for E_b since they are the same or $\tau = K_A \cdot Ø \cdot I_a$

It is to be noted that this expression for the torque induced in a motor is similar to the voltage induced in a DC Generator except that the speed **ω** in the DC Generator is replaced by the Armature current I_a . The constant K_A is same and is given by $K_A = (ZP/2\pi a)$

In general, the torque τ in the DC motor will depend on the following 3 factors:

1. ***The flux Ø in the machine***
2. ***The armature current I_a in the machine***
3. ***The same constant K_A representing the construction of the machine***

Types of DC Motors and their output (or terminal) Characteristics:

There are three important types DC Motors: DC separately excited, Shunt and Series motors. We will explain their important features and characteristics briefly.

The terminal characteristic of a machine is a plot of the machine's output quantities versus each other.

For a motor, the output quantities are shaft torque and speed, so the terminal characteristic of a motor is a plot of its output ***torque versus speed.*** (Torque/Speed characteristics)

They can be obtained from the Motor's Induced voltage and torque equations we have derived earlier plus the Kirchhoff's voltage law around the armature circuit and are again given below for quick reference.

The internal voltage generated in a DC motor is given by:	$E_b = K_a \cdot \Phi \cdot \omega$
The internal Torque generated in a DC motor is given by:	$\tau = K_a \cdot \Phi \cdot I_a$
KVL around the armature circuit is given by :	$V_T = E_b + I_a \cdot R_a$

Where	Φ	= Flux per pole	....	Webers
	I_a	= Armature current	....	Ampere

V_{Ts} = Applied terminal Voltage .... Volts

R_a = Armature resistance .... Ohms

ω = Motor speed .... Radians/sec

E_b = Armature Back EMF .... Volts

K_a = **(ZP/2πa)** : Motor Back EMF/Torque constant

From the above three equations we get the relation between Torque and speed as:

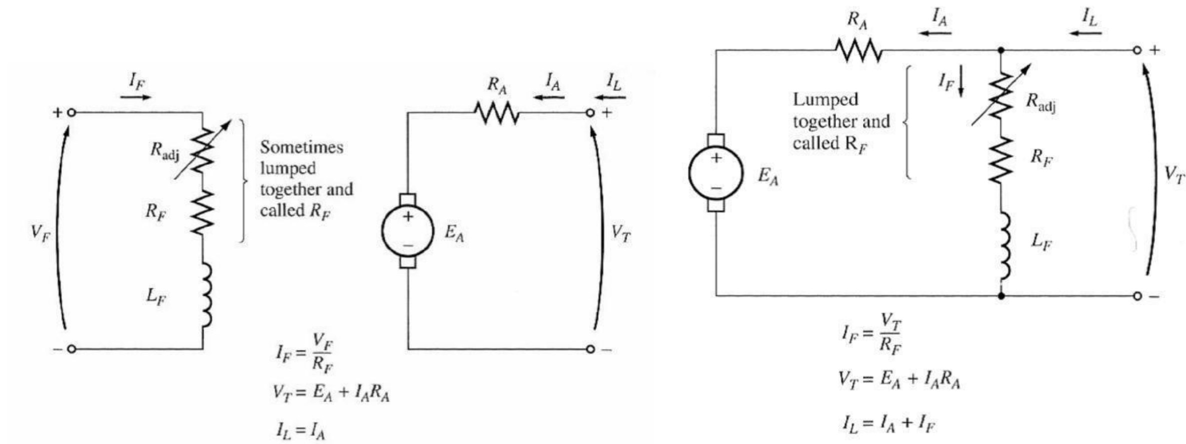
$$\omega = (V_T / K_a.\ \Phi\ ) -- (R_a / K_a.\ \Phi).\ I_a$$

$$\omega = (V_T / K_a.\ \Phi\ ) -- [R_a / (K_a.\ \Phi)^2].\tau$$

We will use this equation in different types of motors and obtain their *Torque vs. Speed* characteristics.

DC separately excited and Shunt Motors:

The Equivalent circuits of DC separately excited and Shunt Motors along with their governing equations are shown in the figure below.



(a) Separately Excited (b) Shunt

Fig: Equivalent circuit of DC separately excited and Shunt Motors

In a separately excited DC motor the field and armature are connected to separate voltage sources and can be controlled independently. In a shunt motor the field and the armature are connected to the same source and cannot be controlled independently. When the supply voltage to a motor is assumed constant and is same to the field and armature circuits, there is no practical difference in behavior between these two machines. Unless otherwise specified, whenever the behavior of a shunt motor is described, it would be same as that of a separately excited motor.

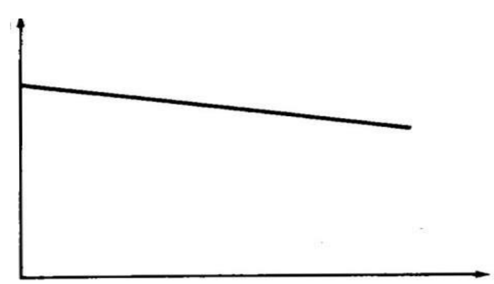
In both their cases, with a constant field current the field flux can be assumed to be constant and then **(K_a. Φ)** would be another constant **K**. Then the above Torque speed relations would become:

$$\omega = V_T / K -- (R_a / K).\ I_a$$

$$= V_T / K -- [R_a / (K)^2].\tau$$

This equation is just a straight line with negative slope. The resulting Speed/ Torque Characteristics of a DC Separately Excited /Shunt Motor for a rated terminal voltage and full field current are shown in the figure below. It is a drooping straight line.

Speed (ω)



Torque (T)

Fig: Speed/ Torque Characteristics of a DC Separately Excited/Shunt Motor

The no load speed is given by the Applied armature terminal voltage and the field current. Speed falls with increasing load torque. The speed regulation depends on the Armature circuit resistance. The usual drop from no load to full load in the case of a medium sized motor will be around 5%. Separately excited motors are mostly used in applications where good speed regulation and adjustable speed are required.

DC Series Motor:

The equivalent circuit of a DC Series motor is shown in the figure below.

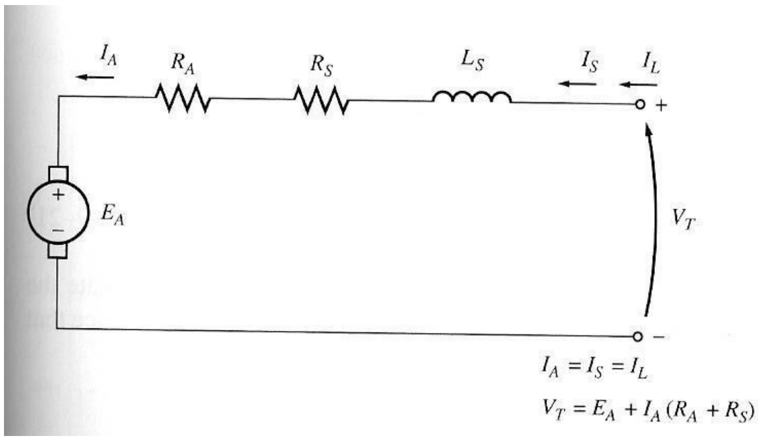


Fig: Equivalent Circuit of a DC Series Motor

In a series motor the field current and armature current are same and hence the field flux is directly dependent on the armature current. Hence during the initial i.e unsaturated region of the magnetization characteristic the flux **Φ** can be assumed to be proportional to the armature current.

Then $\mathbf{\Phi = K_f.I_a}$

And using this value in the first basic motor relation given earlier we get:

$$\tau = K_a.\ \Phi.I_a = K_a.\ K_f.I_a^2$$

$$\tau = K_{af}.I_a^2 \quad (\text{ where } K_{af} = K_a.K_f\ )$$

Substituting the above two values of **Φ** and **τ** in the second basic motor equation

$$\omega = (V_T\ /\ K_a.\ \Phi\ ) -- [R_a/ (K_a.\ \Phi)^2].\tau$$

We get

$$\omega = V_T\ /\ K_a.\ K_f.I_a -- [R_a/\ (K_a.\ K_f\ .I_a)^2].K_{af}.I_a^2$$

$$\omega = V_T\ /\ K_{af}.I_a -- [R_a/\ (K_{af}\ .I_a)^2].K_{af}.I_a^2$$

$$\omega = V_T\ /\ K_{af}.I_a -- [R_a/\ (K_{af})]$$

From the relation $\mathbf{\tau = K_a.\ \Phi.I_a = K_a.\ K_f.I_a^2}$ **we get** $\mathbf{I_a = \sqrt{\tau}/K_{af}}$ and substituting this in the above equation $\mathbf{\omega = V_T\ /\ K_{af}.I_a -- [R_a/\ (K_{af})]}$

We get

$$\omega = *V_T\ /\sqrt{(\ K_{af}.\tau)}] -- [R_a/(K_{af})]$$

Where $\mathbf{R_a}$ is now the sum of armature and field winding resistances and $\mathbf{K_{af} = K_a.K_f}$ is the total motor constant. The Speed-Torque characteristics of a DC series motor are shown in the figure below.

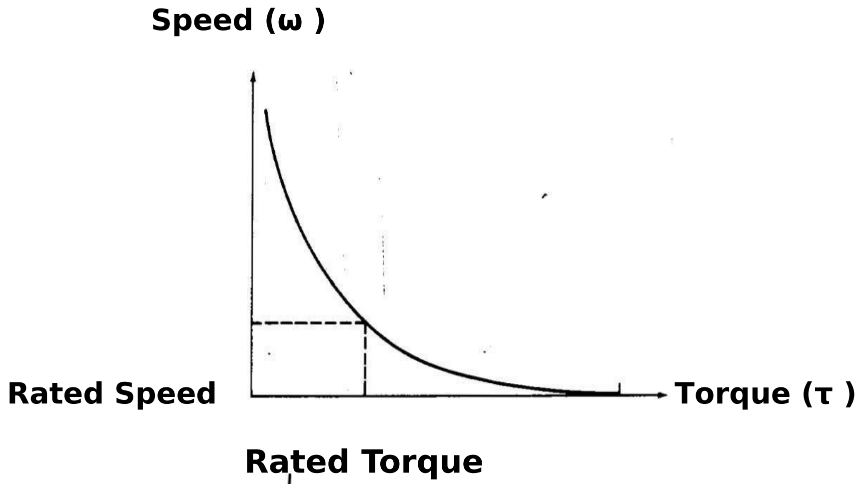


Fig: Speed-Torque characteristics of a DC series motor

Series motors are suitable for applications requiring high starting torque and heavy overloads. Since Torque is proportional to square of the armature current, for a given increase in load torque the increase in armature current is less in case of series motor as compared to a separately excited motor where torque is proportional to only armature current. Thus during heavy overloads power overload on the source power and thermal overload on the motor are kept limited to reasonable small values. According to the above Speed torque equation, as speed varies inversely to the square root of the Load torque, the motor runs at a large speed at light load. Generally the electrical machine's mechanical strength permits their operation up to about twice their rated speed. Hence the series motors should not be used in such drives where there is a possibility for the torque to drop down to such an extent that the speed exceeds twice the rated speed.

Speed control of DC Shunt Motor:

There are two basic methods of DC Shunt Motor speed control

- Armature Voltage Control (AVC) and
- Flux control

Armature Voltage Control (AVC):

This method involves changing the voltage applied to the armature of the motor without changing the Voltage applied to the field. This is possible with a Separately excited DC Motor only and not with DC Shunt Motor. So first we shall explain for a DC separately excited motor and extend the same logic to a shunt Motor. If the armature terminal Voltage V_T is increased, then the I_A will rise since $[\ I_A = (V_T \uparrow - E_b)/R_A]$. As I_A increases, the induced torque $\tau = K_a.\ \Phi.I_a \uparrow$ **increases**, making $\tau_{ind} > \tau_{load}$, and the speed of the motor increases.

But, as the speed increases, $E_b = K_a.\ \Phi.\omega \uparrow$ increases, causing the armature current I_A to decrease since $[\ I_A = (V_T - E_b \uparrow)/R_A]$. This decrease in I_A decreases the induced torque, causing τ_{ind} to become equal to

τload at a final higher steady state rotational speed. Thus we can see that an increase in Armature voltage results in a higher speed and the resulting Speed Torque characteristics with **AVC** is shown in the figure below.

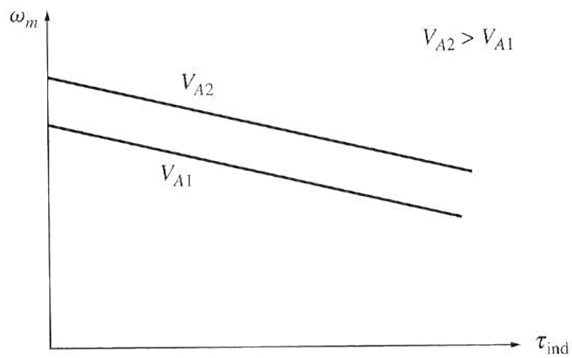


Fig: The effect of armature voltage speed control

In the case of a DC Shunt motor since changing the voltage applied to the armature of the motor without changing the Voltage applied to the field is not possible a Variable resistance is introduced in series with the Armature which results in a reduction in the Armature current I_A. Effectively reduction of Armature current is equivalent to reduction in Armature voltage as seen in the above logic. Hence we get the same type of Speed control as shown in the figure above except that the characteristic with $\mathbf{V_{A2}}$ represents the nominal rated speed and that with $\mathbf{V_{A1}}$ represents with additional resistance introduced in series with the Armature. With this method speed control is possible but speed can only be reduced from the rated or nominal speed. Even for a separately excited DC Motor it can provide speed control

below Base speed only because armature voltage cannot exceed the rated value.

Flux control:

Another method of Shunt motor speed control is to change the flux in the field. In a shunt motor Field current and hence field flux cannot be changed without changing the armature volage.Hence flux control in Shunt motor is achieved by changing the Field resistance. If the field resistance increases, then

the field current decreases $(\mathbf{I_F\downarrow = V_T/R_F\uparrow})$, and as the field current decreases, the flux also decreases . A decrease in flux causes an instantaneous decrease in the internal generated voltage (back emf) $\mathbf{E_b\downarrow = K_a.\ \Phi\downarrow.\omega}$ which causes a large increase in the machine's armature current since,

$\mathbf{I_A \uparrow = (V_T - E_B\downarrow)/R_A}$

The induced torque in a motor is given by $\mathbf{\tau ind = K_a.\ \Phi\downarrow.I_a\uparrow}$ **.** Here since the flux in this machine decreases while the current $\mathbf{I_A}$ increases, which way does the induced torque change?

From practical data it can be seen that for a given decrease in flux the increase in armature current is much higher and hence the increase in current predominates over the decrease in flux caused by the Increase in field resistance.

so, $\mathbf{\tau ind}$ > $\mathbf{\tau load}$, the motor **speeds up**.

However, as the motor speeds up, $\mathbf{E_b}$ rises, causing $\mathbf{I_A}$ to fall. Thus, induced torque $\mathbf{\tau ind}$ too drops, and finally $\mathbf{\tau ind}$ equals $\mathbf{\tau load}$ at a higher steady-state speed than the original speed .

The Speed Torque characteristics with change in Field Resistance are shown in the figure below.

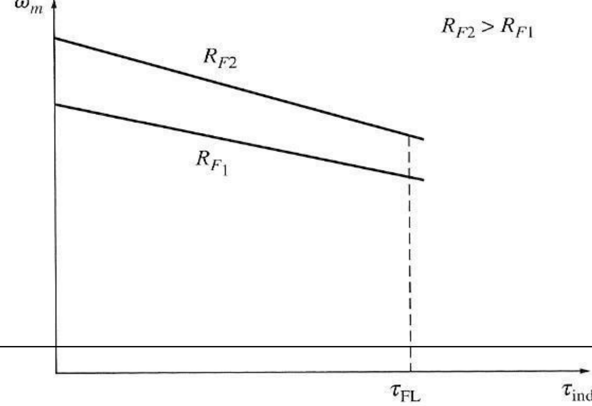


Fig: Shunt Motor Speed control with Flux control (Change in field resistance)

Field Flux Control can be employed For speeds above Base speed only as to achieve speeds below base speeds field current has to be increased beyond its rated value which is not permitted. In a normally designed motor the maximum speed can be twice the rated speed and in specially designed motors it can be up to six times the rated speed.

$P_F = I_F^2 R_F$

Losses and efficiency (η) of DC Machines:

DC Generators convert Mechanical power into Electrical power and DC Motors convert Electric power Into mechanical power. In either case not all the input power is converted into output power.In the process of conversion some power is lost. The following are the important components of losses.

1. Electrical or Copper Losses (I^2R Loss):Current flow through the resistance of Armature and Field cols

give rise I^2R losses and since the coils are normally made up of copper these losses are called Copper losses.

Armature copper loss: $P_A = I_A^2 R_A$
Field copper loss:

2. Core Losses: They are the hysteresis and eddy current losses occurring in the Armature and Field cores

3. Mechanical Losses: They are associated with the mechanical effects and they are mainly Friction and windage losses. Friction losses are losses caused by the friction in the bearings of the machine and windage losses are due to the friction between the moving parts of the machine and the air flow in the machine housing .

4. Stray Losses: They are other miscellaneous losses that cannot be grouped into any of the above categories.

Efficiency:

The efficiency of a DC Machine is defined as $\boldsymbol{\eta = (P_{out}/P_{in}). 100\ \%}$

Efficiency calculations of Generator:

- If $\mathbf{I_L}$ is the load current supplied by the Generator at a terminal voltage of $\mathbf{V_T}$ then the output power is given by

The armature current	$\mathbf{I_A = I_L + I_F}$
Armature copper loss	$\mathbf{P_A = I_A^2 R_A}$
Field copper loss	$\mathbf{P_F = I_F^2 R_F}$

-
- Total losses losses. $= I_A^2 R_A + I_F^2 R_F + W_C$ where W_C is the sum of the core losses and stray
-
- Therefore Input $= P_{out}$ + **Total losses** $= P_{out} + I_A^2 R_A + I_F^2 R_F + W_C$

Hence $\eta = (P_{out}/P_{in}).\ 100\ \% = (V_T.I_L)/(V_T.I_L + I_A^2 R_A + I_F^2 R_F + W_C).100\%$

Efficiency calculations of Motor:

If I_L is the line current taken by the Motor at a terminal voltage of V_T then the input power is given by

The losses are same as in the Generator

Therefore output $P_{out} = P_{in}$ -- **Total losses** $= P_{in} - (I_A^2 R_A + I_F^2 R_F + W_C)$

Hence $\eta = (P_{out}/P_{in}).\ 100\ \% = [\{P_{in} - (I_A^2 R_A + I_F^2 R_F + W_C)\}/(V_T.I_L)]\ .\ 100\%$

Of these losses ($I_F^2 R_F + W_C$) **are** called constant losses P_C since they are almost independent of load since the terminal voltage is constant . The armature copper losses i.e. ($I_A^2 . R_A$) is called the variable loss and is dependent on the load. The variable loss varies approximately as the square of load. We say approximately since loss varies as the square of the armature current and not as the square of the load current. Hence if we know the loss at full load the loss at half load, onefourth load etc can be calculated.

Maximum efficiency:

The condition for maximum efficiency is developed by differentiating the expression for efficiency as a function of load current and equating it to zero since the variable losses are dependent on the load current. The condition is obtained as:

$$\textbf{Constant losses } P_C = \textbf{Variable losses } (I_L^2 R_A) \textbf{ or } I_L = \sqrt{(P_C / R_A)}$$

Swinburne's test:

This is a test to determine the efficiency of any DC Machine (Motor or Generator) without conducting the actual test at the required load. The test is conducted just at no load and the constant losses are found out when the machine is running as a motor. Then the efficiency is found out by calculating the variable losses at the required load. This method is formulated by Sir James Swinburne and hence it is called Swinburne's test.

The machine is run as a motor on no load at normal terminal voltage V_T , at normal speed and the line current I_{NL} &field current I_F are measured.

- Then the no load armature current I_A
- Variable losses on no load

  be measured these losses can be calculated)
- Input to the motor = $V_T . I_{NL}$ = **Total losses** (Since the machine is on no load there is no output. i.e. the entire input power on no load goes as losses.)
- Therefore constant losses P_C = **(Total losses - Variable losses)** = $(V_T . I_{NL}) - (I_A^2 . R_A)$

Using these constant losses P_C , the efficiency of the machine can be estimated at any other load when working either as a Motor or as a Generator.

Working as a Generator delivering a load current of I_L amperes at a terminal voltage of V_T volts:

Power output = $V_T . I_L$

- Armature current $I_A = I_L + I_F$ (IF is same as obtained in the No load test)
- Variable loss = $I_A^2 R_A$ (R_A is obtained from the no load test or from Machine data)
- **Efficiency = (output/Input) = [output/(output+Total losses)] =** $(V_T . I_L)/(V_T . I_L + I_A^2 R_A + P_C)$

  (P_C is calculated and obtained from the No load test)

Working as a Motor drawing a load current of I_L amperes from a supply terminal voltage of V_T volts:

- Power in put = $V_T . I_L$
- Armature current I_A =
- $I_L - I_F$ (I_F is same as obtained in the No load test)
- Variable loss = $I_A^2 R_A$ (R_A is obtained from the no load test or from Machine data)

Efficiency = (output/Input) = [(Input-Total losses)/ input] =[V_T. I_L-($I_A^2R_A$+P_C)]/(V_T. I_L)

(P_C is calculated and obtained from the No load test)

***Advantages of Swinburne's test*:**

- This is a very simple to determine the efficiency of the machine at any load just by conducting the no load test.
- The power required is very less compared to the direct full load test.

***Disadvantages of Swinburne's test*:**

- This test can be done on Shunt machines only.
- The speed and flux are assumed constant. But the speed will fall with loading. Fall in speed results in lesser friction and windage losses. Change in flux will change the core losses.
- The temperature of the machine changes while running on load. Hence the assumption that **R_A** is same as that of the No load test is not correct.
- These reasons contribute to the difference in the efficiency obtained from the Swinburne's test and actual load test.

Important concepts and Formulae

- **Voltage generated in a DC machine: E_A = (Ø ZN/60). (P/a) and in terms of angular speed ω :**

  $E_A = K_a\, Ø\, \omega$ where $K_a = ZP/2\pi a$

- **Torque generated in a DC machine : $\tau = K_a. \Phi.I_a$**
- **Speed control with armature voltage control is possible only below the rated or nominal speed (also known as base speed).**
- **Speed control with flux control is possible only above the base speed**
- **The efficiency of a DC Machine is defined as η = (P_{out}/P_{in}). 100 %**

  ***Efficiency of Generator*:**

  η **= (P_{out}/P_{in}). 100 % = (V_T.I_L)/(V_T.I_L + I_A^2 R_A + I_F^2 R_F +W_C).100%**

  Efficiency of Motor:

  η **= (P_{out}/P_{in}). 100 % = [{V_T.I_L -(I_A^2 R_A + I_F^2 R_F +W_C)}/ (V_T.I_L)] . 100%**

- **The condition for maximum efficiency :**

  Constant losses P_C = Variable losses ($I_L^2R_A$) or $I_L = \sqrt{(P_C/ R_A)}$

llustrative Examples

Ex.1: Calculate the e.m.f. generated by a 6 pole DC Generator having 480 conductors and driven at a speed of 1200 RPM. The flux per pole is 0.012 webers. (a) When the machine is lap wound (b) When the machine is wave wound

Solution: We know that the e.m.f. generated by a DC Generator is given by

E_A = (Ø ZN/60) (P/a) where

Ø Flux per pole

(webers) = 0.012Wb

Z Total number of conductors on the armature = 480 a The number of parallel paths **=** No of Poles P (= 6) when Lap wound and = 2 when wave wound

N Speed of rotation of the machine (RPM) = 1200 RPM

P The number of poles = 6

(a) For Lap wound machine a = P = 6

E_a = [(0.012 x 480 x 1200) / 60] [6/6] **= 115.2Volts**

(b) For wave wound machine a = 2

E_a = [(0.012 x 480 x 1200) / 60] [6/2] **= 345.6Volts**

Ex.2 : A 50 Kw ,250 V shunt generator operates at 1500 RPM .The armature has 6 poles and is lap wound with 200 turns. Find the induced e.m.f and the flux per pole at full load given that the armature and the field resistances are 0.01 Ω and 125 Ω respectively.

Solution:

Output line current = Output power / Line voltage = 50 x 1000 / 250 = 200 A

Field current = Line Voltage / Field resistance = 250 / 125 = 2 A

Armature current in a shunt generator: = $I_l + I_f$ = 200 + 2 =202 A

Induced e.m.f E_a : = Line Voltage + Armature drop ($I_a R_a$ drop)

= 250 + 202 x 0.01 = **252.02 V**

But we know that armature voltage in terms of the basic machine parameters is also given by

E_A = (Ø ZN/60) (P/a) where

Ø c: Flux per pole (webers) = To be determined

Z : Total number of conductors on the armature has = Number of turns x 2 (since each turn two conductors) = 200 x 2 = 400 a **:** The number of parallel paths **=** No of Poles P (= 6) (since Lap wound)

N : Speed of rotation of the machine (RPM) = 1500 RPM

P **:** The number of poles = 6

∴ **Ø = (E_A x 60 x a/ ZNP)** = 252.02 x 60 x 6 / 400 x 1500 x 6 = **0.025202 Wb**

Ex.3: A shunt generator connected in parallel to supply mains is delivering a power of 50 Kw at 250 V while running at 750 RPM. Suddenly its prime mover fails and the machine continues to run as a motor taking the same 50 Kw power from 250 V mains supply. Calculate the speed of

the machine when running as a motor given that R_a = 0.01 Ω, R_f = 100 Ω and brush drop is 1 V per brush.

Solution:

First let us calculate the Voltage generated by the machine while running as a generator under the given conditions:

Output line current = Output power / Line voltage = 50 x 1000 / 250 = 200 A

Field current = Line Voltage / Field resistance = 250 / 100 = 2.5 A

Armature current : $I_l + I_f$ = 200 + 2.5 = 202.5 A

Induced e.m.f E_a := Line Voltage + Armature drop (I_aR_a drop)+ Brush drop(two brushes)

= 250 + 202.5 x 0.01 + 2 x1 = 254.025 V

Next let us calculate the Voltage generated by the machine while running as a motor under the given conditions :

Input line current = Input power / Line voltage = 50 x 1000 / 250 = 200 A

Field current = Line Voltage / Field resistance = 250

/ 100 = 2.5 A Armature current : $I_l - I_f$ = 200 - 2.5 =

197.5 A

Induced e.m.f or back e.m.f E_b : = Line Voltage - Armature drop(I_aR_a drop) - Brush drop(two brushes)

= 250 – 197.5 x 0.01 - 2 x1 = 246.025 V

We know that the voltage induced in the machine is proportional to the speed i. e

Generator armature voltage is proportional to Generator speed : E_a N_G and similarly

Motor back e.m.f is proportional to Motor speed : $E_b \quad N_M$

Hence $E_a N_G = E_b \quad N_M$ or $\mathbf{N_M}$ = $(E_b E_a) N_G$ = (246.025 / 254.025) x 750 = **726 RPM**

Ex.4: A 500 V shunt motor with R_f = 250 Ω and R_a = 0.2 Ω runs at 2500 RPM taking a current of 25 A from the mains supply . Calculate the resistance to be added to the armature circuit to reduce the speed to 1500 RPM keeping the armature current constant.

Solution:

First let us calculate the back e.m.f developed by the motor in the given first set of conditions:

Field current I_f = Rated terminal voltage / R_f = 500 / 250 = 2 A

Armature current $I_a = I_l - I_f$ = (25 – 2) = 23 A

Back e.m.f E_b = $V_T - I_a R_a$ = 500 – 23 x 0.2 = 495.4 V

We know that the back e.m.f is proportional to the speed

$E_{b1} / E_{b2} = N_1 / N_2$ i.e 495.4 / E_{b2} = 2500/1500

E_{b2} = 495.4 x 1500 / 2500 = 297.24 V

But we also know that E_{b2} = $V_T - I_a R_{a2}$ (Since the terminal voltage and the armature current remain the same)

297.24 = 500 – 23 x R_{a2} from which we get R_{a2} = (500 – 297.24) /23 = 8.82 Ω

Hence the new resistance to be added into the armature circuit = 8.82- 0.2 = **8.62 Ω**

Ex.5: A DC shunt motor takes 22 A from 250 V supply. R_a = 0.5 Ω , R_f = 125 Ω. Calculate the resistance required to be connected in series with the armature to halve the speed (a) when the load torque is constant (b) When the load torque is proportional to the square of the speed **olution :**

First let us calculate the speed of the motor when the load current I_l is 22 A :

Field current I_f = Rated Terminal voltage / Field resistance = 250/125 = 2 A

Armature current $I_a = I_l - I_f$ = 22 -2 = 20 A

Back e.m.f $E_b = V_T - I_a R_a$ = 250 – 20 x 0.5 = 240 V

(a) *we have to find out the New R_a when the speed is halved with torque maintained constant* :

We know that Torque $T = K_a. Ø.I_a$. In this case since change is only in the armature resistance

field current and hence flux Ø remains the same. Further since the torque is maintained constant the armature currents are also equal and hence $I_{a1} = I_{a2}$ = 20 A

We also know that $E_b = K_a. Ø.ω$. As already explained, K_a. Ø remains same and hence when the speed is halved the back e.m.f also gets halved.

Hence E_{b2} = 120 V = $V_T – I_aR_{a2}$ i.e 250 – 20 x R_{a2} = 120 V i.e R_{a2} = (250 -120)/20 = 6.5Ω Hence the **Resistance to be added to halve the speed** = $R_{a2} - R_a$ = 6.5-0.5 = **6.0 Ω**

(b)Next we have to find out the New R_a when the speed is halved when torque is proportional to square of speed.

When the torque is proportional to the square of the speed $τ_1 = K ω_1^2$ and $τ_2 = K ω_2^2$

∴ $τ_1 / τ_2 = K ω_1^2 / K ω_2^2 = ω_1^2 / ω_2^2 = (1/0.5)^2 = 4$

But Torque is also proportional to the product of flux (and hence field current)and Armature current. Here field circuit is not disturbed and hence the field current is same. Using this relation we can find out new armature current I_{a2}

∴ $τ_1 / τ_2 = K \times I_f \times I_{a1} / K \times I_f \times I_{a2} = I_{a1} / I_{a2} = 4$ i.e $I_{a2} = I_{a1} / 4$ = 20/4 =5 A

Next using the relation between the speeds and the back emfs we can find out the armature resistance to be added.

$\omega_1/\omega_2 = 2$ and also

$\omega_1/\omega_2 = E_{b1}/E_{b2} = 240 / (250 – 5 \times R_{a2})$ i.e $250- 5R_{a2} = 240/2 = 120$

From which we get $R_{a2} = (250-120)/5 = 26\ \Omega$

Finally **Resistance to b added** is = 26-0.5 = **25.5 Ω**

Ex.6: A 250 V DC series motor takes 40 A and runs at 1000 RPM. Find the speed at which it runs if its torque is halved. Assume that the motor is operating in the unsaturated region of its magnetization. R_f = 0.25 Ω R_a = 0.25 Ω

First we will use the relation between torque and armature current and get the back e.mf when the torque is halved :

In a DC motor we know that the torque is proportional to $Ø.I_a$. In the case of a series DC motor flux is proportional to the armature current itself since $I_f = I_a$. Hence in a series motor $\boldsymbol{\tau \propto I_a^2}$

Hence $\boldsymbol{\tau_1/\tau_2 = I_{a1}^2 / I_{a2}^2} = 2$ (Since torque is halved)

$\boldsymbol{I_{a1} / I_{a2}} = \sqrt{2}$

I_{a1} = 40 A and $I_{a2} = 40/\sqrt{2}$ = 28.28 A

E_{b1} = 250 - 40 (0.25 + 0.25) = 230 V and E_{b2} = 250 - 28.28 (0.25 + 0.25) = 235.86 V

Next we will use the relation between back emf and speed and get the speed when the torque is halved:

We know that $E_{b1} = K_a Ø_1N_1$ and $E_{b2} = K_a Ø_2N_2$. But since the flux is proportional to I_a the relations become $E_{b1} = K I_{a1}N_1$ and $E_{b2} = K I_{a2}N_2$ where K is a new constant. Hence $E_{b1}/ E_{b2} = K I_{a1}N_1 / K I_{a2}N_2 = I_{a1}N_1 / I_{a2}N_2$ and $N_2 = (I_{a1} / I_{a2}) (E_{b2}/ E_{b1}) N_1$

Substituting the above values we get $N_2 = \sqrt{2}$ (235.86/230) 1000 = **1450 RPM**

Ex.7: A 500 V DC shunt motor runs at 1900 RPM taking an armature current of 150 A. The armature resistance is 0.16 Ω. Find the speed of the motor when a resistance is inserted in the field circuit which reduces the field current to 80 % and the armature current is 75 A.

Solution:

We know that the back e.m.f of a DC motor is proportional to the Flux and speed. And in the unsaturated region of the magnetization region the flux in turn is proportional to the field current. So Back e.m.f is proportional to field current and speed. We will find out the new speed by calculating the back e.m.fs [from the relation ($E_b = V_T - I_a R_a$)]and using the above proportionality relation in both the conditions as below.

$E_{b1} = V_T - I_{a1} R_a$ = 500 – 150 x 0.16 = 476 V and is equal to $K_a.Ø_1. N_1$

$E_{b2} = V_T - I_{a2} R_a$ = 500 – 75 x 0.16 = 488 V and is equal to $K_a . 0.8Ø_1. N_2$

$\therefore$ $476 / 488 = K_a . Ø_1 . N_1 / K_a . 0.8Ø_1 . N_2$

And **N_2** $= (488 / 476)(N_1 / 0.8) = (488 / 476)(1900 / 0.8) =$ **2435 RPM**

Ex.8: A DC shunt motor having a full load efficiency (**η **)of 85 % takes a line current of 27A from

250 Volts mains on full load. If $R_a = 0.5\Omega$ and $R_f = 125\ \Omega$, find the constant losses, load current for maximum efficiency and the maximum efficiency.

Solution:

Input power at full load = Full load current x Rated voltage = 250 x 27 = 6750 W

Output power = Input power x η = 6750 x 0.85 (η = 85%) = 5737.5 W

Hence Total losses = Input power – Output power = 6750 – 5737.5 = 1012.5 W We know that Total losses = Variable losses ($I_a^2 R_a$) + constant losses.

I_f = Rated Terminal Voltage/ Field resistance = 250 / 125 = 2 A

For the Shunt motor armature current $I_a = I_l - I_f$ = 27 – 2 = 25 A

Variable losses = $I_a^2 R_a = 25^2 \times 0.5$ = 312.5

Constant losses = Total losses – Variable losses = 1012.5 – 312.5 = 700 W

We know that the condition for maximum efficiency is: **Variable losses = Constant losses**

i.e. $I_a^2 R_a$ at maximum efficiency = 700

∴ $I_{a\ @max.\ \eta} = \sqrt{700/0.5} = \sqrt{1400}$ = 37.42 A

∴ The load current at maximum η : $I_{l@max.\ \eta} = I_{a@max.\ \eta} + I_f$ = 37.42 + 2 = 39.42

Input power at maximum η = $I_{l\ @max.\ \eta}$ x Rated terminal voltage = 39.42 x 250 = 9855 W

O/P power at maximum η = I/P power at maximum η Total losses = 9855–(700 +700) = 8455 W

(Since variable losses = constant losses = 700)

Maximum efficiency = Out power at maximum efficiency/ Input power at maximum efficiency

= 8455 / 9855 = 0.858 or 85.8 %

Ex.9: A 100 Kw 500 V DC shunt machine when run as a motor on no load at rated speed and voltage takes a line current of 10 A and a shunt field current of 2.5 A . Resistance of the armature is 0.15 Ω. Estimate the efficiency of the DC machine when running as a generator (a) at full load (b) at half full load.

First the constant losses of the Machine are obtained from the data we have when the machine is run as a motor on no load at rated speed and voltage:

Input power on no load = Rated voltage x Input current on no load = 500 x 10 = 5000 W

Field current I_f = 2.5 A

No load Armature current $I_a = I_{l\ no\ load} - I_f$ = (10 – 2.5) = 7.5 A

Variable loss at no load = $(I_{a\ on\ no\ load})^2 \times R_a$ = $7.5^2 \times 0.15$ = 8.4375 W

Constant Losses = (Input power– Variable losses)(on no load) = 5000 - 8.4375 = **4991.56** W

Next we will calculate the efficiency in different conditions:

(a) Running as a generator at full load:

Full load output (line) current = 100 x 1000 / 500 = 200 A

Full load armature current = Full load line current + Field current = 200 + 2.5 = 202.5

A Variable (Armature copper) losses on full load = $I_a^2 R_a = 202.5^2 \times 0.15$ = 6150.94 W

Total losses @ full load = Constant Losses + Variable losses on full load =

4991.56 + 6150.94 = 11142.5 W

Efficiency at full load (Working as Generator) = Output / Input = Output / Output + Total losses
@ full load = 100000 / 100000 + 11142.5 = 0.8997 or **89.97 %**

(a) Running as a generator at half full load:

Half Full load output (line) current = 50 x 1000 / 500 = 100 A

Half Full load armature current = Half Full load line current + Field current =

100 + 2.5 = 102.5 A Variable (Armature copper) losses on half full load = I_a^2

R_a = 102.5^2 x 0.15 = 1575.94 W

Total losses @ half load = Constant Losses + Variable losses on half load = 4991.56 + 1575.94

= 6567.5 W

Efficiency at half full load (Working as Generator) = Output / Input = Output / Output + Total losses @ full load = 50000 / 50000 + 6567.5 = 0.8839 or **88.39 %**

Ex.10: A 500 V DC shunt machine takes 5A when running light (on no load) at rated speed and rated voltage as a motor. Calculate the out output power and efficiency when the machine is run as a Motor and taking an Input current of 80 A. Calculate the line current at which the efficiency is maximum and the value of maximum efficiency. R_a = 0.2 Ω and R_f = 250 Ω

***First the constant losses of the Machine are obtained from the data we have when the machine is run as a motor on no load at rated speed and voltage*:**

Input power on no load = Rated voltage x Input current on no load

= 500 x 5 = 2500 W Field current I_f = Rated voltage / R_f = 500 / 250 = 2 A

No load Armature current $I_a = I_{l\ no\ load} - I_f$ = (5 - 2) = 3 A

Variable loss at no load = $(I_{a\ on\ no\ load})^2$ x R_a = 3^2 x 0.2 = 1.8 W

Constant Losses = (Input power– Variable losses)(on no load) = 2500 – 1.8 = **2498.2** W

Next we will calculate the output power and efficiency when the Machine is running as a motor and taking an input current of 80 A :

Armature current = Line current (Input Current) - Field current = 80 - 2 = 78 A

Variable (Armature Copper) losses (with armature current of 78A) = $I_a^2 R_a$

$=78^2 \times 0.2 = 1216.8$ W Total losses = Constant Losses + Variable losses at 80 A line current = 2498.2 + 1216.8 = 3715 W Input Power = 500 x 80 = 40000 W

Out Put Power at 80 A line current (Working as Motor) = Input Power – Total losses at 80 A line current = 40000 -3715 = 36285 **W**

Efficiency at 80 A line current (Working as Motor) = Output Power /Input Power

= 36285/40000 = 0.9071 or **90.71 %**

Finally we will calculate the line current at which the efficiency is maximum and the value of maximum efficiency:

We know that the condition for maximum efficiency is: **Variable losses = Constant losses**

i.e. $I_a^2 R_a$ (variable Losses) at maximum efficiency = **2498.2** W ∴ = √2498.2 /0.2 = √12491 = 111.76 A

∴**The line current at maximum η** : $I_{l@max.\ \eta} = I_{a@max.\ \eta} + I_f$ = 111.76 + 2 = **113.76 A** Input power at maximum η = $I_{l\ @max.\ \eta}$ x Rated terminal voltage = 113.76 x 500 = 56880 W

O/P power at maximum η = I/P power at maximum η – Total losses = 56880 – (2498.2 + 2498.2) = 51883.6 W

(Since variable losses = constant losses = 2498.2)

Maximum efficiency = Out power at maximum efficiency/ Input power at maximum efficiency = 51883.6 /56880 = 0.9122 or **9122 %**

Previous year's Question papers:

May 2011:

5)What are the different types of dc generators? Show the connection diagrams and load characteristics of each type. [15]

6.a) Explain why a dc series motor should never run unloaded.

b) A 200V, 14.92kW, dc shunt motor when tested by Swinburne's method gave the following test results.

Running light: Armature current of 6.5 A and field current = 2.2A

With armature locked: I_a =70A when potential difference of 3V was applied to the brusher.

Estimate efficiency of motor when working under full load. [5+10]

State the principle of operation of a dc generator and derive the expression for the emf generated. [15]

b)A 4 pole, 500V dc shunt motor has 700 wave connected armature conductors. The full load armature current is 60 A and the flux per pole is 30mWb. Calculate the full load speed if the motor armature resistance is 0.2Ω and brush drop is 1V per brush. [7+8]

Explain in detail the construction and principle of operations of DC generators. [15]

6.Discuss in detail the different methods of speed control of a dc motor. [15]

c) A 6 – pole dc shunt generator with a wave – wound armature has 960 conductors. It runs at a speed of 500 rpm. A load of 20Ω is connected to the generator at a terminal voltage of 240V. The armature and field resistances are 0.3Ω and 240Ω respectively. Find the armature current, the induced emf and flux per pole. [15]

6.Sketch the speed – load characteristics of a dc shunt, series and compound motors. Account for the shape of the above characteristic curves. [15]

May 2012:

5.(a) Name the main parts of a DC machine and state the materials of which each part is made of and explain clearly the reasons to select these materials.

(b) A Certain wave wound DC generator running at a speed of 300rpm is to generate an induced emf of about 535V, the ux per pole being 0.055 Wb. Determine the number of poles, if the number of conductors is 650.

7. (a) Explain why Swinburnes test cannot be used to determine the efficiency of DC series motor?

(b)A 4 pole series motor has 944 wave-connected armature conductors at a certain load. The flux per pole is 34.6 mWb and the total mechanical torque developed is 209 N-m. Calculate the line current taken by the motor and the speed at which it will run. The applied voltage is 500 V and total motor resistance is 3.ohms

(b) Draw the circuit model of a DC shunt generator and write the relationship of currents and voltages.

(a) \All general requirements of the electric traction are fulfilled by DC series motors compared to other DC motors". Justify with related equations and characteristics.

(b) A 250V, 4-pole wave wound DC series motor has 888 conductors on its armature. It has armature and field resistance of 0.88ohms .The motor takes a current of 80A. Determine

i) Speed

ii) Gross torque developed if it has a ux per pole of 28 mwb. [7+8]

2. Compare DC generator and DC motor from principle of operation point of view and mention the application of each machine?

(a) A series wound motor runs normally. The field coils are all connected in series. Estimate the speed and current taken by the motor, if the coils are reconnected in two parallel groups of two in series. The load torque increases as the square of the speed. Assume that flux is directly proportional to the current and ignore the losses.

(b)A 220V motor has an armature circuit resistance of 0.6. If the full load armature current is 20A and the no load armature current is 5A, find the change in back e.m.f from no-load to full-load.

(a) With a neat sketch, explain how the direction of rotation of DC motor can be reversed?

(b) Derive the standard torque equation of DC motor from first principles.

Model papers:

g) Define DC generator and DC motor? (2Marks)

h) A 6 – pole dc shunt generator with a wave – wound armature has 800 conductors. It runs at a speed of 600 rpm. A load of 10Ω is connected to the generator at a terminal voltage of 220V. The armature and field resistances are 0.4Ω and 200Ω respectively. Find the armature current and the induced EMF. (3Marks)

8. a) Explain the principle of operation and operation of DC generators.

b)A 4 – pole dc shunt generator with a wave – wound armature has 960 conductors. It runs at a speed of 500 rpm. A load of 20Ω is connected to the generator at a terminal voltage of 240V. The armature and field resistances are 0.3Ω and 240Ω respectively. Find the armature current, the induced emf and flux per pole.

(OR)

9. a) Derive the torque equation of a dc motor.

b)A 4 pole, 500V dc shunt motor has 700 wave connected armature conductors. The full load armature current is 60 A and the flux per pole is 30mWb. Calculate the full load speed if the motor armature resistance is 0.2Ω and brush drop is 1V per brush.

h)A 250V, 4-pole wave wound DC series motor has 888 conductors on its armature. It has armature and field resistance of 0.88ohms .The motor takes a current of 80A. Determine.

i) Speed.

ii) Gross torque developed if it has a flux per pole of 28 mw. (3Marks)

8. a) What are the different types of dc generators? Show the connection diagrams and load characteristics of each type.

b)A short shunt compound generator delivers a load current of 30A at 220V and has a armature, series and shunt field resistances are 0.05 ohms, 0.03 ohms and 200 ohms respectively Calculate the induced EMF and armature current. Allow 1V per brush contact drop.

(OR)

9. a) Explain why Swinburne’s test cannot be used to determine the efficiency of DC series motor?

b) A 4 pole series motor has 944 wave-connected armature conductors at a certain load. The flux per pole is 34.6 mWb and the total mechanical torque developed is 209 N-m. Calculate the line current taken by the motor and the speed at which it will run. The applied voltage is 500 V and total motor resistance is 3 ohms.

h) A Certain wave wound DC generator running at a speed of 300rpm is to generate an induced emf of about 535V, the ux per pole being 0.055 Wb. Determine the number of poles, if the number of conductors is 650. (3Marks)